Proper Heavy Quark Potential
from the Thermal Wilson Loop

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In collaboration with T. Hatsuda and S. Sasaki

see also arXiv:0910.2321

Lattice QCD confronts experiments
日独セミナー・三島・2010年11月05日

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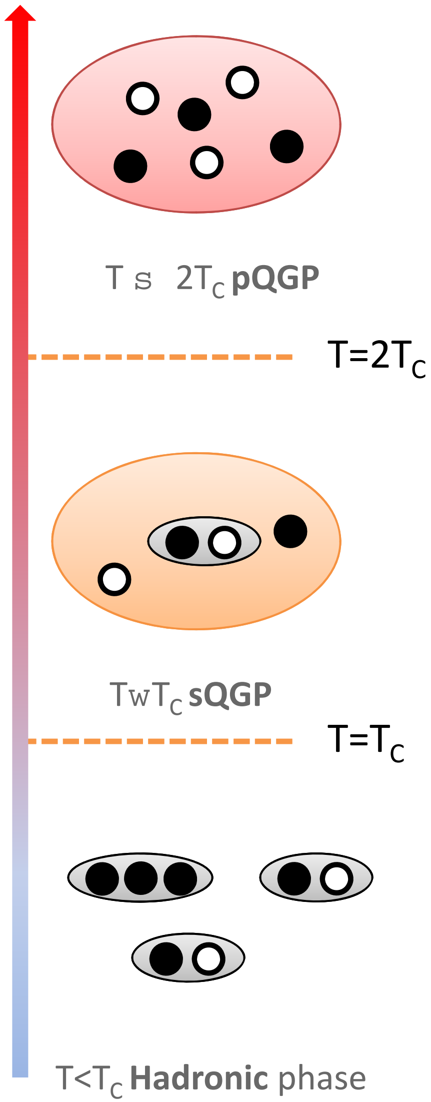
Motivation



- Hadronic Thermometer for the QGP
  - Matsui & Satz (`86): Complete melting of J/ψ at $T=T_C$
- Convenient Separation of Scales
  - Large mass allows a non-relativistic description:

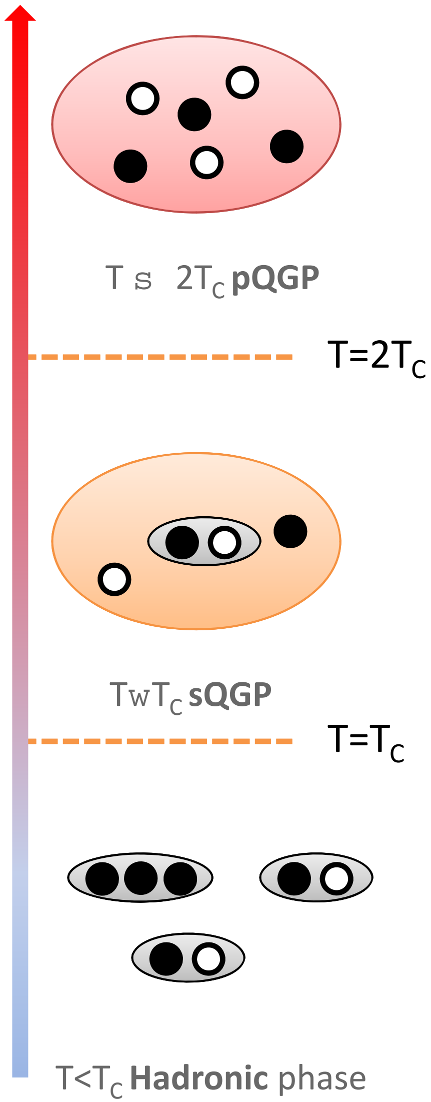
$$\frac{v_{Q\bar{Q}}}{c} \ll 1 \quad \text{or} \quad \left(\frac{p_{Q\bar{Q}}}{m_{Q\bar{Q}}} \ll 1\right)$$

  - Temperatures are comparatively small (at least for $b\bar{b}$)

$$\frac{T_{med}}{m_{Q\bar{Q}}} \ll 1 \qquad \left(\frac{\Lambda_{QCD}}{m_{Q\bar{Q}}} \ll 1\right)_{T=0}$$

- Derive an effective Schrödinger equation for Heavy Quarkonium at finite temperature
  - At m=∞ separation distance of constituents is external parameter

Previous Studies



Hard Thermal Loop — Laine et al. **JHEP 0703:054,2007**

- Real-time formulation based on the forward correlator
- Resummed **perturbation** theory: applies at very high T
- Quarkonia is transient: $V(R) = V_{DS}(R) + iV_{LD}(R)$

Potential Models — eg. Petreczky et al. **arXiv:0904.1748**

- **Ad-hoc** identification with Free Energies in Coulomb Gauge

$$V(R) = \Delta F_{Q\bar{Q}}(R) = -\beta^{-1}\log\left[\langle P(R)P^\dagger(R)\rangle\right] \in \mathbb{R}$$

- Challenges: Gauge dependence, Entropy contributions

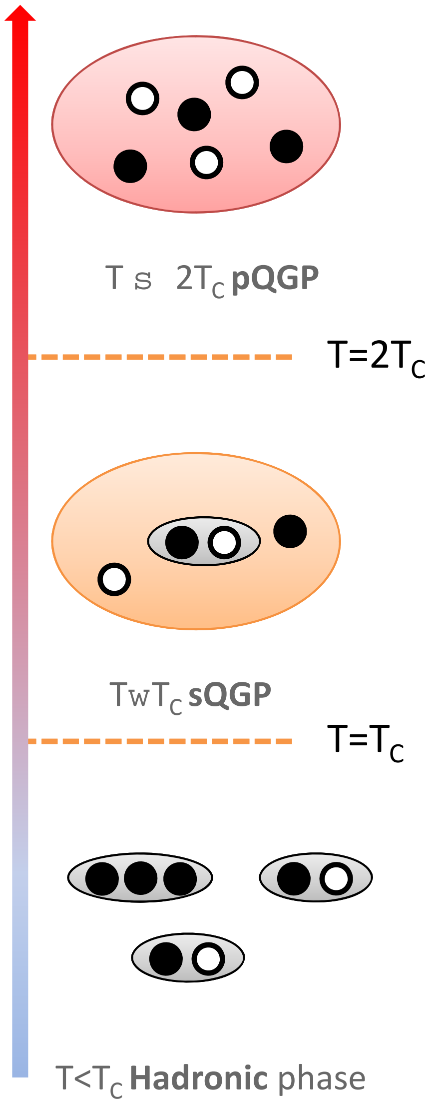
T=0 NRQCD & pNRQCD — Brambilla et al. **Rev.Mod.Phys. 77 (`05)**

- Expansion in 1/m (Minkowski time):

$$V_{m=\infty}(R) = \lim_{t\to\infty} \frac{i}{t}\log\left\langle \mathrm{Tr}\left(\mathcal{P}_\square \exp\left[\frac{ig}{c}\int_\square dx_\mu A^\mu(x)\right]\right)\right\rangle$$

- Remember: $\psi(t) = \exp[imt]\exp\left[\frac{ig}{c}\int dt A^0(t)\right]$

Previous Studies II



Spectral functions from LQCD Asakawa, Hatsuda **PRL 92 012001,('04)**

- Extract **spectral functions** directly from non-perturbative **Lattice QCD** simulations at any temperature
- Challenging but well understood: **Maximum Entropy Method**
  Asakawa, Hatsuda: Prog.Part.Nucl.Phys.46:459,('01);
- Unexpected: **J/ψ** seems to **survive** up to $T > 1.6T_C$
  Free Energies predict melting at $1.2\ T_C$

This talk:

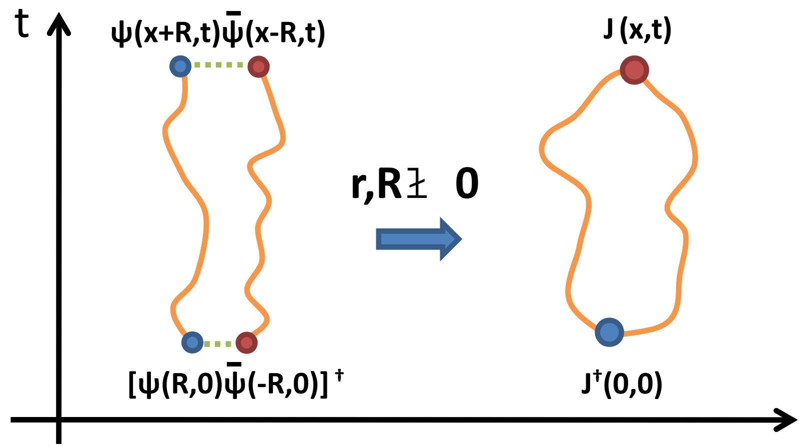
- Combine the systematic expansion of the **effective theory** with the non-perturbative power of **Lattice QCD at T>0**
- We propose a **gauge invariant** and **non-perturbative** definition of the static proper in-medium heavy quark potential based on the **thermal Wilson loop**.
- **Numerical results** for the proper potential

Starting point

- The point-split mesic correlator

$$M(x,y) = \bar{\psi}(x)\overbrace{\Gamma}^{\text{Gamma matrix}}\underbrace{W(x,y)}_{\text{spatial Wilson line}}\psi(y)$$

$$D^{>}(t,\mathbf{R}) = \langle M(t,\mathbf{R})M^{\dagger}(0,\mathbf{R})\rangle$$



- m ‹ T: thermal fluctuations cannot excite QQbar pairs
- Imaginary time boundary condition becomes irrelevant for QQbar
- Quantum mechanical description becomes possible: Schrödinger equation
- Only naturally gauge invariant quantity: current - current correlator (Dilepton production)

$$\left[-i\partial_{t'} + \sum_i \frac{p_i^2}{2m} + V(R)\right]D^{>}(r,t,R,t') = 0 \Rightarrow \lim_{r,R\to 0} D^{>}(r,R,t,t') = \langle J(x,t')J^{\dagger}(0,t)\rangle$$

From first principles QCD

- Reduce the degrees of freedom for the heavy fermions

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{q}_l\left(i\gamma_\mu D^\mu - mc\right)q_l + \bar{\psi}\left(i\gamma_\mu D^\mu - mc\right)\psi$$

- Foldy-Tani-Wouthuysen Transform: Expansion up to inverse rest mass $1/mc^2$

$$\mathcal{L}_{NRQCD} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left[(i\gamma^0 D_0 - mc) + \frac{g}{2mc^2}\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}B^i + \frac{1}{2mc}D_i^2\right]\psi$$

  - Path integral **measure** is **unchanged**

$$D^>_{NRQCD} = \int \mathcal{D}[\text{Med}]\left[\mathcal{T}\int \mathcal{D}[\psi,\bar{\psi}]\Gamma_{\alpha\beta}\Gamma_{\gamma\delta}W_{ab}W^\dagger_{cd}\psi^\beta_b(y')\bar{\psi}^\gamma_c(y)\psi^\delta_d(x)\bar{\psi}^\alpha_a(x')e^{iS_{NRQCD}[\psi,\bar{\psi},A]}\right]e^{iS_{Med}[A,\psi_l,\bar{\psi}_l]}$$

  - No coupling of upper and lower components of Dirac 4-spinor -> **No creation/annihilation**
  - Heavy fermions do not appear in virtual loops: Heavy quark **determinant** can be **neglected**

2x2 sub matrix

$$D^>_{pNRQCD} = \int \mathcal{D}[\text{Med}]\mathcal{T}\, W(x,y)\boxed{G}\,\boxed{S(y,y')}\,\boxed{\bar{G}}W^\dagger(x',y')\boxed{S^\dagger(x,x')}\boxed{e^{iS_{Med}}}$$

Temperature dependence

NRQCD heavy quark greens function (2x2)

QM path integral picture

- Express Greens functions as quantum mechanical path integral   Barchielli et. al. PRB296 625, 1988

$$i\partial_t S(x,x') = \left[\frac{1}{2m}\left(-i\nabla - \frac{g}{c}A(x)\right)^2 + gA^0(x) - \frac{g}{mc}\sigma_i B^i(x)\right]S(x,x') \qquad \& \qquad S(x,x')|_{t=t'} = \delta^3(x-x')$$

$$\Downarrow$$

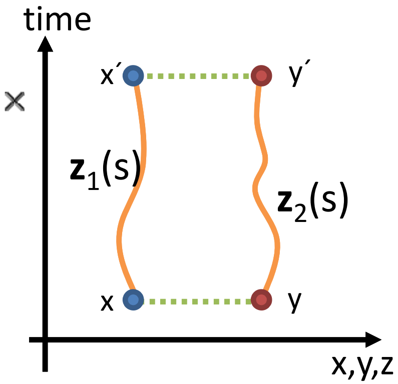
$$S(x,x') = \int_x^{x'} \mathcal{D}[z,p]\mathcal{T}\exp\left[i\int_t^{t'} dt\left(p(t)\dot{z}(t) - \frac{1}{2m}\left(p(t) - \frac{g}{c}A(z(t),t)\right)^2 - gA^0(z(t),t) + \frac{g}{mc}\sigma_i B^i(z(t),t)\right)\right]$$

- Combine the paths from both constituents:

$$D^{>} = \exp[-2imc^2 t]\int \mathcal{D}[z_1,p_1]\int \mathcal{D}[z_2,p_2]\exp\left[i\int_t^{t'} ds \sum_i \left(p_i(s)\dot{z}_i(s) - \frac{1}{2m}p_i(s)^2\right)\right]\times$$

$$\left\langle \frac{1}{N}\mathrm{Tr}\left[\mathcal{P}_{\mathcal{C}}\exp\left[\frac{ig}{c}\boxed{\oint_{\mathcal{C}} dx^\mu A_\mu(x)} + \sum_j \frac{ig}{mc^2}\int_{z_j} dx^\mu \sigma_j^i F_{i\mu}(x)\right]\right]\right\rangle_A$$

This is not just the Wilson loop: fluctuating paths



- To obtain a potential term in the time Hamiltonian operator for $D^>$ we need:

$$\langle \mathrm{Tr}[\exp[\oint A + \int_i \frac{\sigma_i}{m}F]]\rangle \equiv \exp\left[i\int_t^{t'} ds\; U(z_1(s), z_2(s), p_1(s), p_2(s), s)\right]$$

The proper static potential

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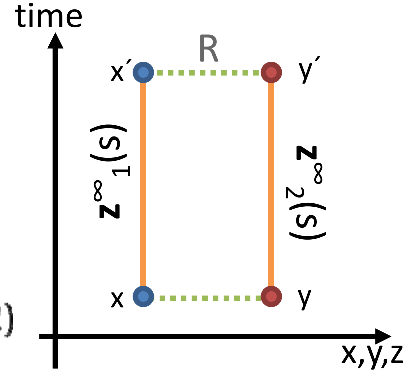
- Expansion in the momenta: $$\int_t^{t'} ds\, U(z(s),p(s),s) = \int_t^{t'} ds\left(\boxed{u(z,s)|_{p=0}} + w_n^i(z,s)|_{p=0}\,\frac{p_n^i(s)}{mc} + \dots\right)$$
  Barchielli et. al. PRB296 625, 1988

  - For m ± — : $$i\log\left\langle \frac{1}{N}\mathrm{Tr}\left[\mathcal{P}_C\exp\left[\frac{ig}{c}\oint_C dx^\mu A_\mu(x)\right]\right]\right\rangle\Big|_{p_{1,2}=0} = \int_t^{t'} ds\, u(R,s)$$

    Real-time thermal Wilson loop



  - Use the spectral function (Fourier transform) to obtain:

$$u(R,t) = i\,\partial_s \log\left[WL(s,R)\right]\Big|_{s=t} = \frac{1}{\int d\omega e^{-i\omega t}\rho_\square(\omega,R)}\int d\omega e^{-i\omega t}\,\omega\,\rho_\square(\omega,R)$$

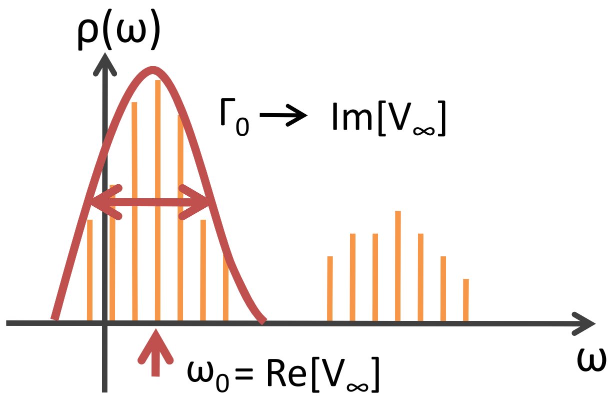
  - The notion of a static potential is valid if the peak structure of ρ(ω,R) is well defined:

$$\int_{-\infty}^{\infty} d\omega\,\omega\, e^{-i\omega t}\rho_\square(\omega,R) \overset{!}{=} V(R)\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\rho_\square(\omega,R)$$

  - Analytically solvable cases: Breit Wigner and Gaussian

$$\Rightarrow \quad V_\infty^{BW}(R) = \omega_0(R) + i\Gamma_0(R)$$

$$\Rightarrow \quad V_\infty^{GA}(R) = \omega_0(R) + i\Gamma_0^2(R)\,t$$



At finite momentum

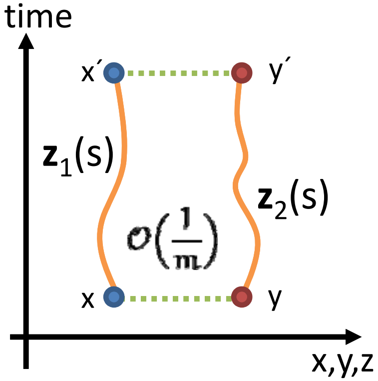
- There is not yet a kinetic term, since the paths collapsed to a straight line

$$i\partial_t D^>(t,R) = \Big[2mc^2 + \mathrm{Re}V_\infty(R) + i\mathrm{Im}V_\infty(R)\Big]D^>(t,R) \qquad \text{at} \quad \mathcal{O}\Big(\frac{1}{m^0}\Big)$$

- Going to next order in p: $\int_t^{t'} ds\, U(z(s),p(s),s) = \int_t^{t'} ds\Big(u(z,s)\big|_{p=0} + \boxed{w_n^i(z,s)\big|_{p=0}}\,\frac{p_n^i(s)}{mc} + \dots\Big)$
  - Note that position and momentum are independent in Hamiltonian formalism
  - To current order: $i\log\Big\langle \frac{1}{N}\mathrm{Tr}\Big[\mathcal{P}_C \frac{\delta}{\delta p(t)} \exp\Big[\frac{ig}{c}\oint_C dx^\mu A_\mu(x)\Big]\Big]\Big\rangle\Big|_{p_{1,2}=0} = 0$
  - Field independent part of the path integral gives kinetic term

- The final result at $\mathcal{O}\Big(\frac{1}{m}\Big)$

$$\boxed{i\partial_t D^>(t,R) = \Big[2mc^2 + \frac{p^2}{2m} + \mathrm{Re}V_\infty(R) + i\mathrm{Im}V_\infty(R)\Big]D^>(t,R)}$$



  - A dynamical Schrödinger equation for the proper complex heavy quark potential

Spetral function from LQCD

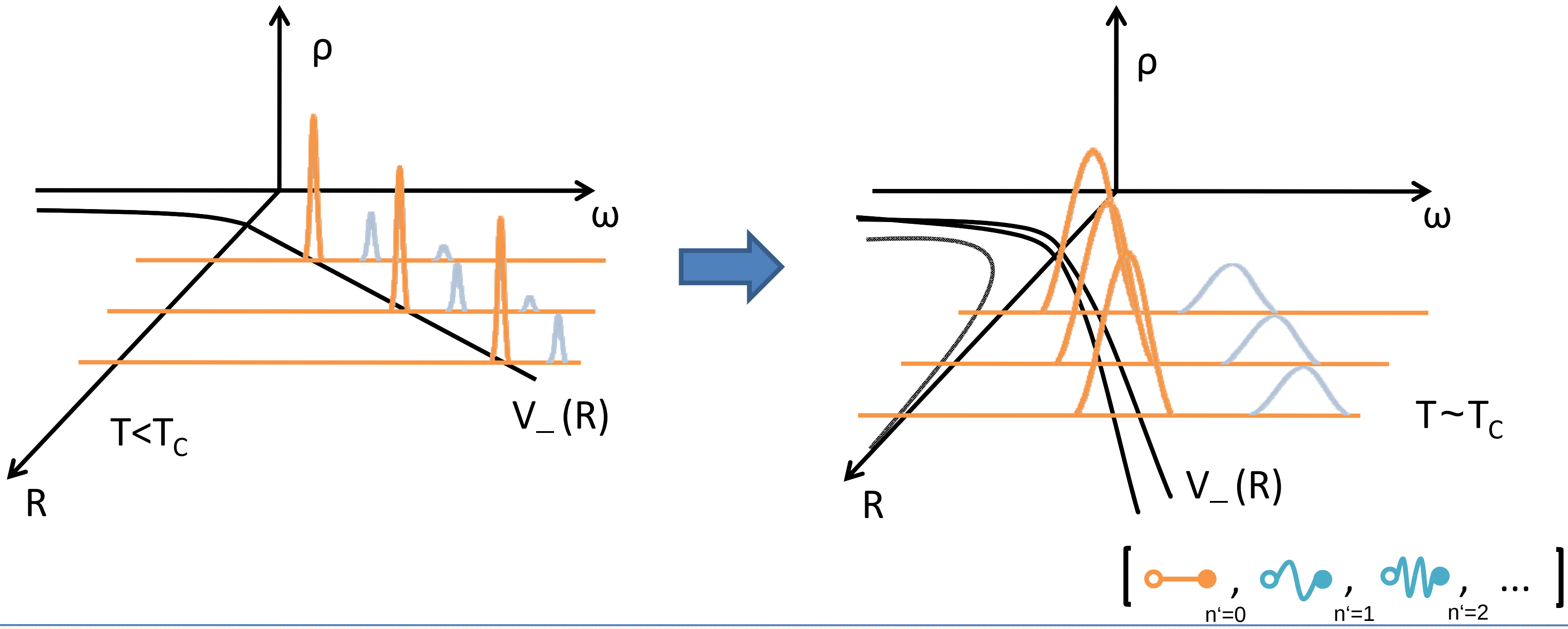
- Definition of the potential requires knowledge about the Wilson Loop spectral function
  - Spectral function connects Euclidean and Minkowski time

$$WL_M(-i\tau,R)=\int_{-\infty}^{\infty} d\omega\, e^{-\omega\tau}\boxed{\rho(\omega,R)}=\boxed{WL_E(\tau,R)}$$

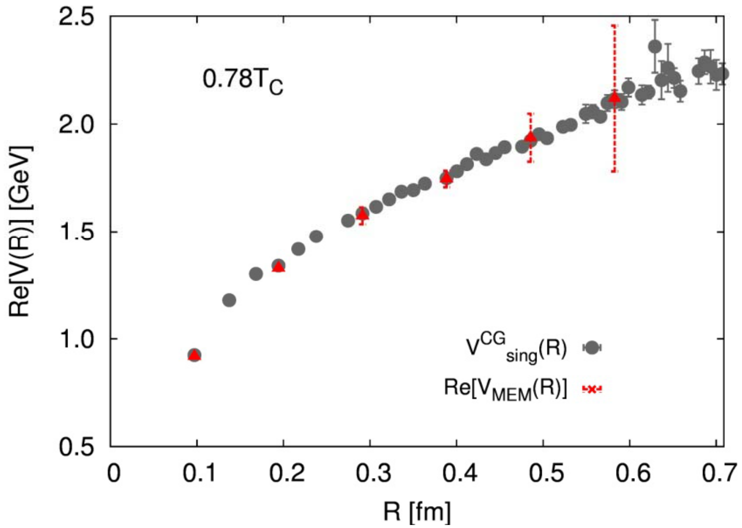
Simulate in LQCD

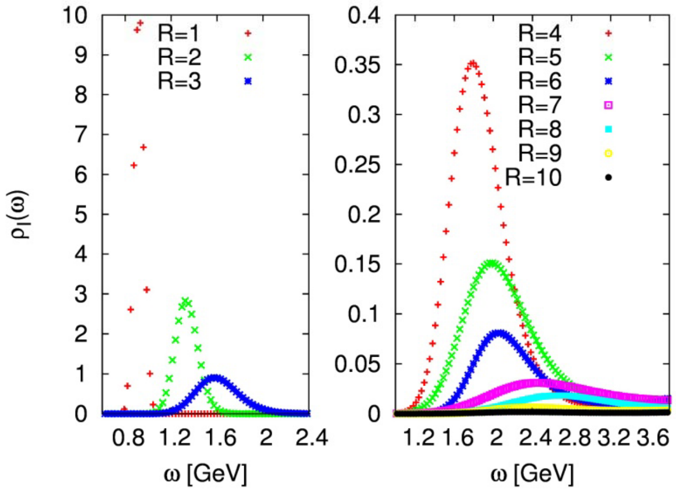
Extraction via MEM

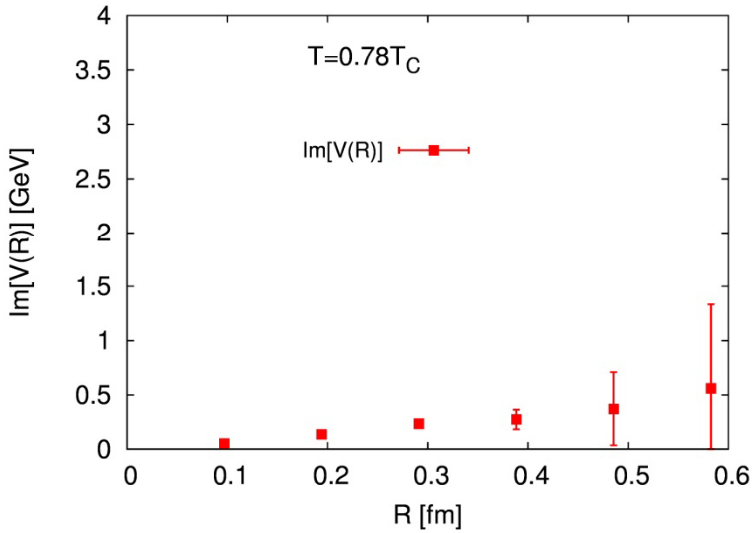
- Exploring the proper potential



- **Confinement region:**







- Real Part coincides with the potential from Free Energies in Coulomb Gauge
- Imaginary part small: possibly artifact from the MEM

Numerical Results $T=2.33T_C$

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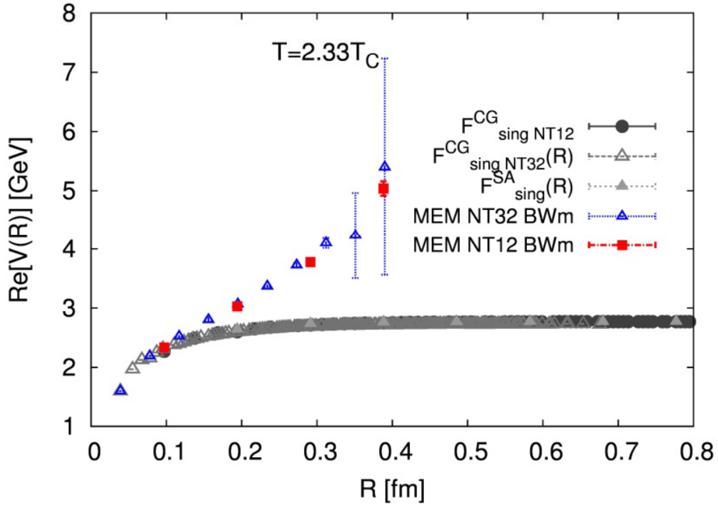
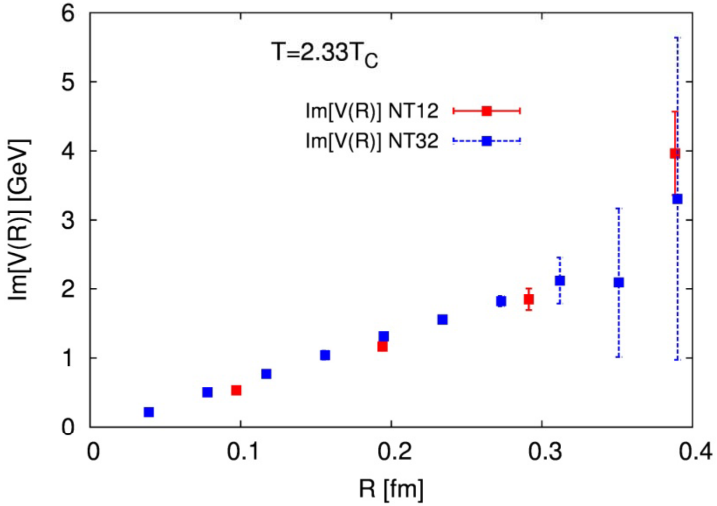
- **Deconfinement region: still s(Q)GP**





Quenched QCD Simulations

MEM Reconstruction (BR, no SVD)

- **Anisotropic Wilson Plaquette Action**
- **NX=20 NT=12 β=6.1 ξ_b=3.2108**
- **Δx = 4Δτ = 0.1fm**
- N_τ=12 I_τ=(0,6.1] and N_τ=32 I_τ=(0,7]
- **N_C= 3500 HB:OR = 1:5 N_{sweep}=200**
- Prior: m_0 $1/(\omega-\omega_0+1)$
- **NX=20 NT=32 β=7.0 ξ_b=3.5**
- **Δx = 4Δτ = 0.04fm**
- **N_C= 1100 HB:OR = 1:5 N_{sweep}=200**

- Real Part much steeper than Free Energies in Coulomb Gauge
- Imaginary part appears to be finite
- Width much rather resembles a Gaussian, not Breit-Wigner
- Only solving the Schrödinger equation with both real and imaginary part can tell us about the survival of heavy quarkonia

Conclusion

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- Present approaches toward an in-medium potential
  - Either perturbative (HTL) or plagued by ambiguities (free energies, internal energies)
- Proper heavy quark potential at finite T:
  - Separation of scales used to derive effective theory for Heavy Quarks: NRQCD
  - In the m⊥ — limit**: Spectral function** of the **thermal Wilson Loop** determines the applicability of a potential picture, i.e. **existence of V_ (R)**
  - In the most naïve case: **peak position** corresponds to **real part**, peak **width** to **imaginary par**t of **V(R)**
- Numerical results for purely gluonic medium:
  - Below T_C: **Proper potential coincides** with potential **from free energies** in Coulomb Gauge
  - Above T_C: Real part of the potential much steeper than free energies potential and **imaginary part increases significantly**
- Future work:
  - Light fermions in the medium: Full QCD simulations
  - Derivation of the $1/m^2$ corrections, the highest possible order for the naive potential picture
  - Solving the full time-dependent Schödinger equation

The End

Thank you for your attention

ご清聴ありがとうございました

Vielen Dank für Ihre Aufmerksamkeit

Direct Spectral properties

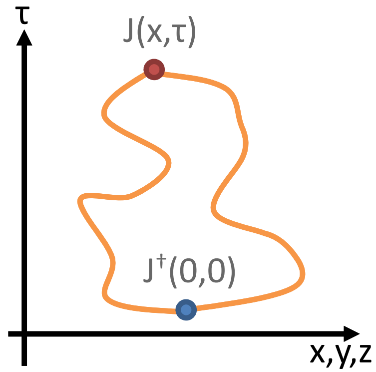
- How to extract spectral information from Euclidean Lattice data: **Maximum Entropy Method**

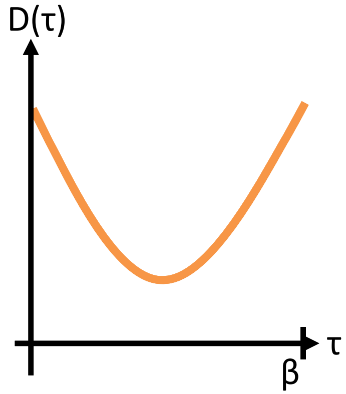
$$J_\mu(\vec{x},\tau)=\bar{\psi}(x)\gamma_\mu\psi(x)$$

$$D^>(\vec{x},\tau)=\mathrm{Tr}\left[e^{-\beta H}\mathcal{T}_\tau J^\mu(\vec{x},\tau)J^\dagger_\mu(\vec{0},0)\right]/Z$$

$$\boxed{D^>(R,\tau)}=\int_0^\infty d\omega\,\frac{e^{-\tau\omega}+e^{(\beta-\tau)\omega}}{1-e^{-\beta\omega}}\boxed{\rho(R,\omega)}$$

O(10) O(1000)





- Ill posed problem: Cannot use usual χ^2 fitting

- Bayes Theorem can help: $P[\rho|Dh]=\dfrac{\boxed{P[D|\rho h]}\boxed{P[\rho|h]}}{P[D|h]}$

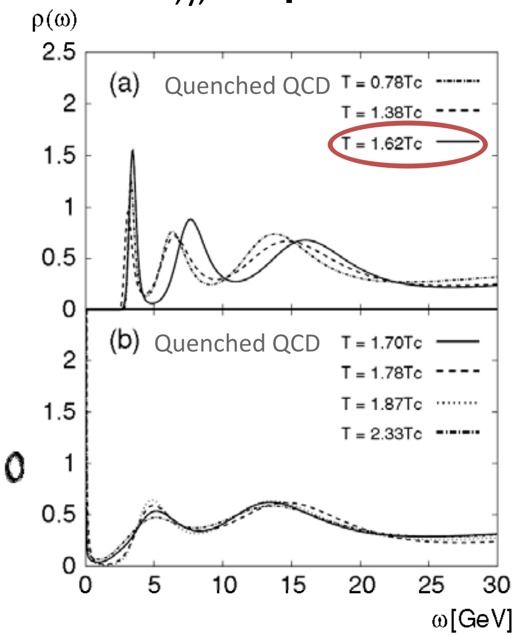
$$\propto \mathrm{Exp}\left[-\frac{1}{2}\sum_{ij}\Big(D(\tau_i)-D_\rho(\tau_i)\Big)C^{-1}_{ij}\Big(D(\tau_j)-D_\rho(\tau_j)\Big)\right]$$

Usual χ^2 fitting term

$$\propto \mathrm{Exp}\left[\alpha\int_{-\infty}^{\infty}\left\{\rho(\omega)-h(\omega)-\rho(\omega)\mathrm{Log}\left(\frac{\rho(\omega)}{h(\omega)}\right)\right\}d\omega\right]$$

Shannon Janes Entropy, makes prior knowledge explicit

$$\frac{\delta}{\delta\rho}P[\rho|Dh]\stackrel{!}{=}0$$



Asakawa, Hatsuda, Nakahara: Prog.Part.Nucl.Phys.46:459-508,2001
see also: Nickel, Annals Phys.322:1949-1960,2007

Heavy Quark Potential Models

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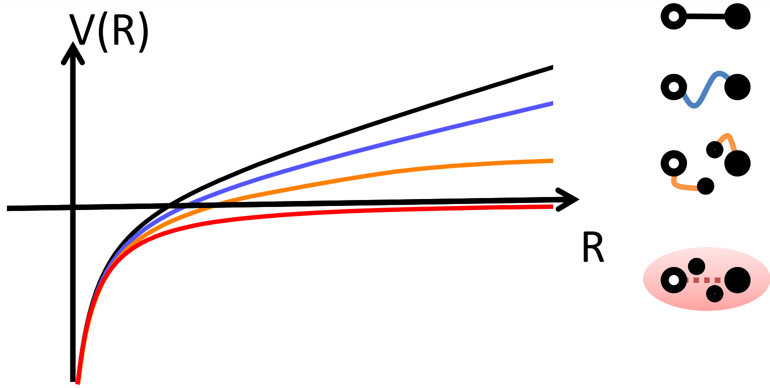
- Phenomenological expectations:

  Confinement: Linear rising below T_C

  String breaking with dynamical fermions

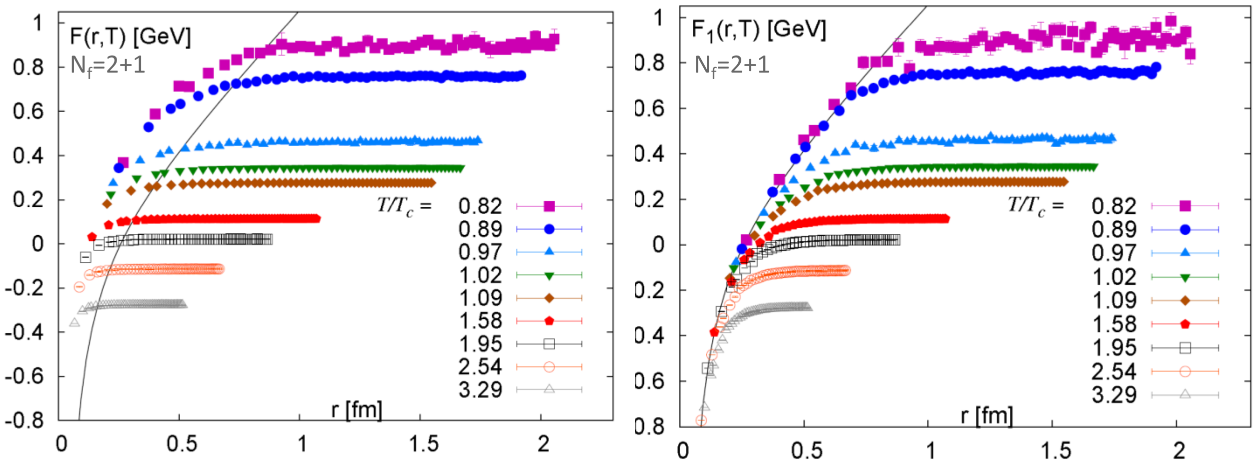
  Screening above T_C due to free color charges



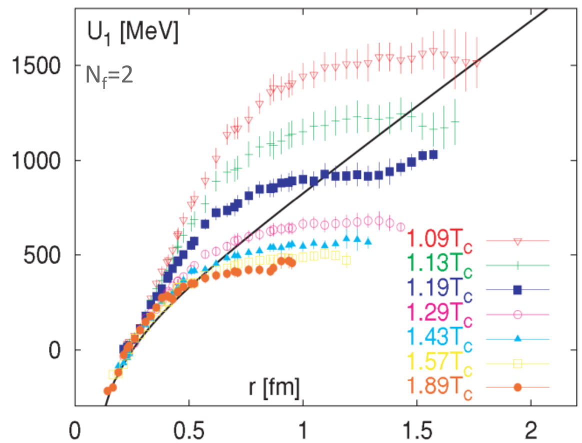
- T=0 result: $$V(R) = -\lim_{\tau\to\infty}\frac{1}{\tau}\mathrm{Log}\left[WL(\tau,R)\right]$$ e.g. Brown, Weisberger PRD20:3239,1979.

- Ad-hoc identification of potentials (No Schrödinger equation was derived)

$$F(R) = -\frac{1}{\beta}\mathrm{Log}\left[\mathrm{Tr}[P(R)]\mathrm{Tr}[P^\dagger(0)]\right] \qquad F^1(R) = -\frac{1}{\beta}\mathrm{Log}\left[\mathrm{Tr}[P(R)P^\dagger(0)]\right] \qquad U^1(R) = F^1(R) - T\left(\frac{\partial F^1(R)}{\partial T}\right)$$



Petreczky, arXiv:1001.5284v2



Satz, J. Phys. G 36 (2009) 064011