Hadronic fluctuations at freeze-out

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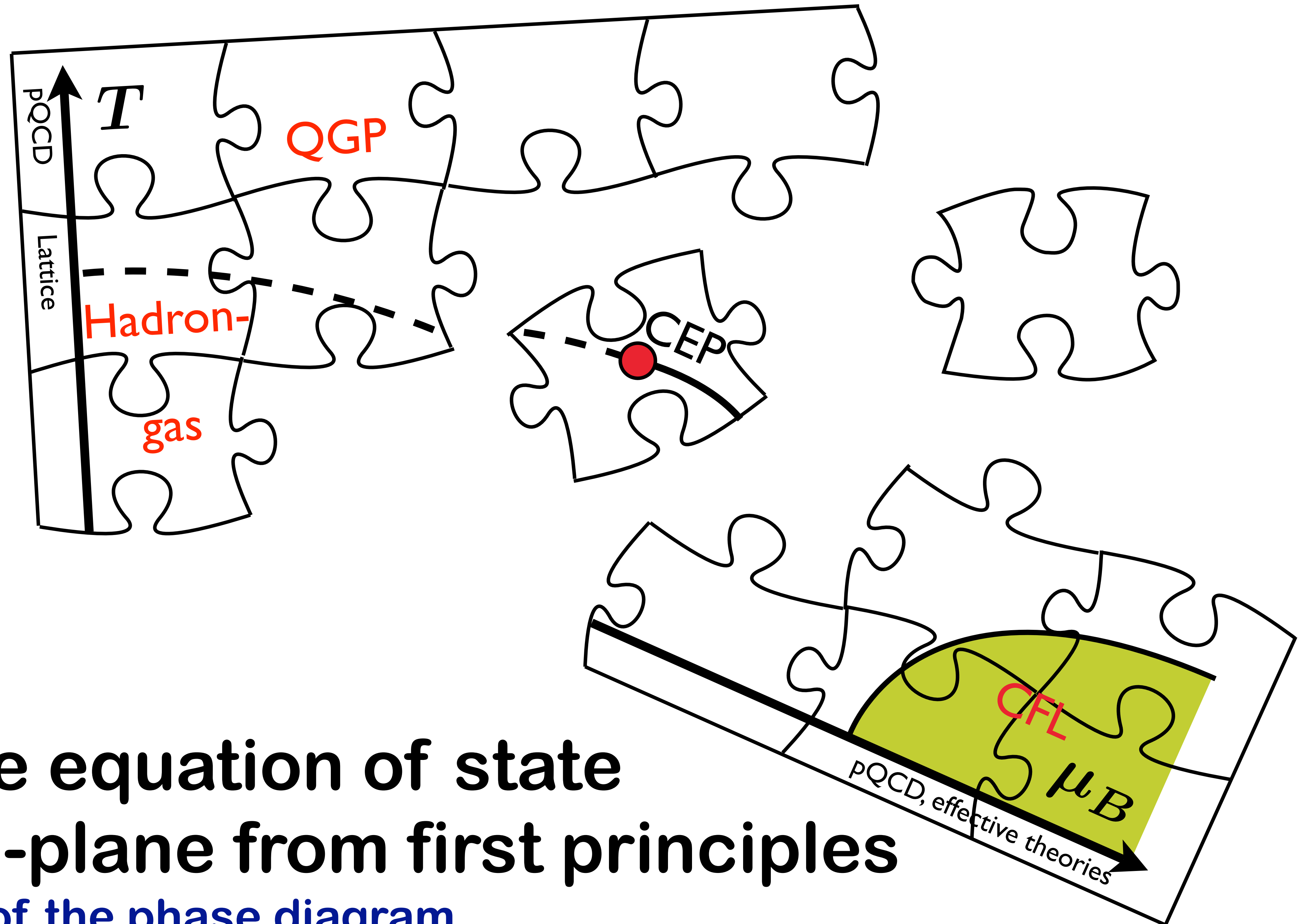
S. Ejiri, F. Karsch, E. Laermann, C. Miao, S. Mukherjee,
P. Petreczky, C.S., W. Söldner, W. Unger

see also: CS, arXiv:1007.5164

Motivation and Overview

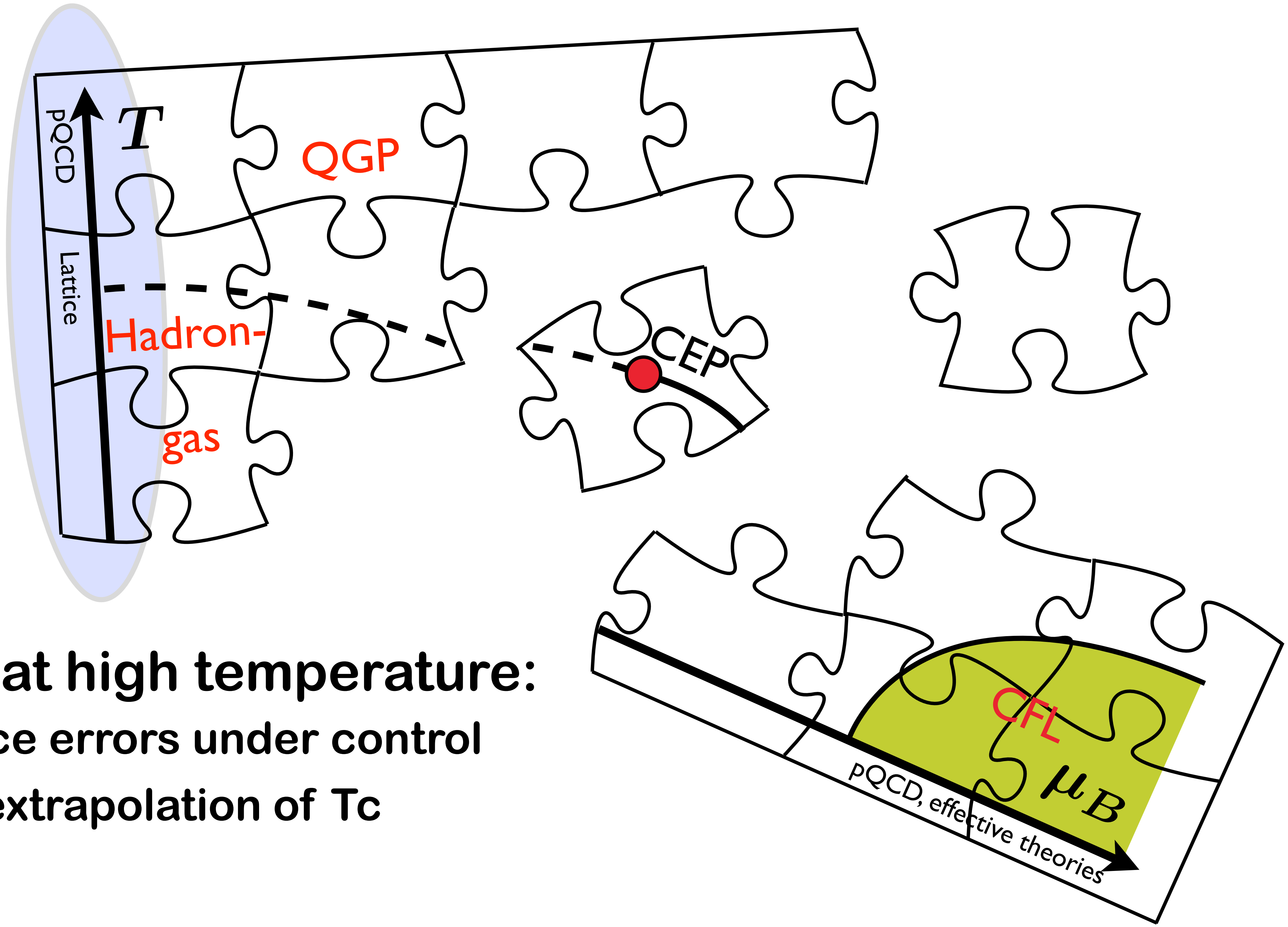


Analyzing the equation of state
in the (T, μ_B)-plane from first principles

$\longrightarrow$ determination of the phase diagram

$\longrightarrow$ understanding underling mechanism of the transition

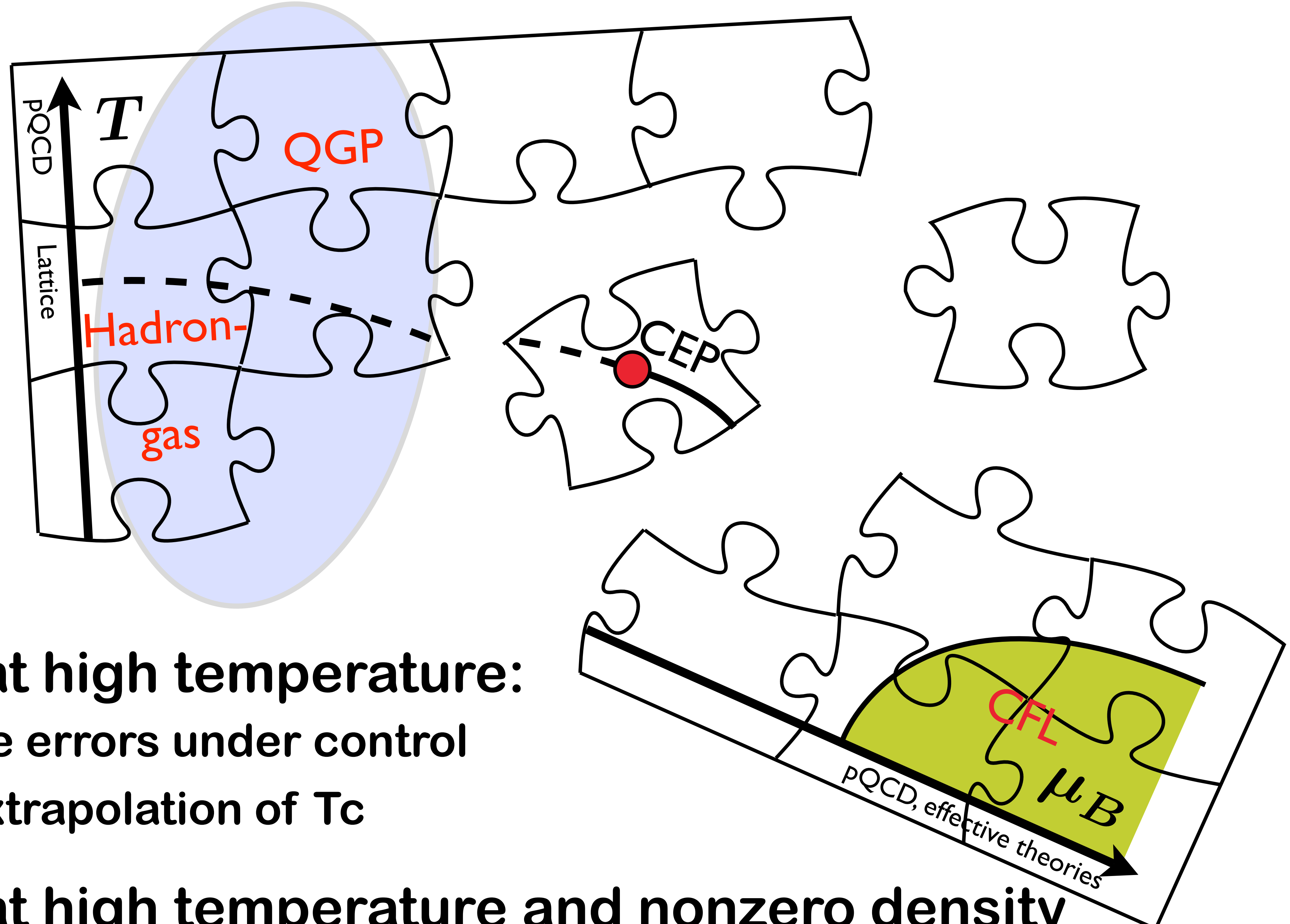
Motivation and Overview



Overview:

★ **Lattice QCD at high temperature:**
- getting lattice errors under control
- continuum extrapolation of Tc

Motivation and Overview



Overview:

★ **Lattice QCD at high temperature:**

- getting lattice errors under control
- continuum extrapolation of Tc

★ **Lattice QCD at high temperature <u>and nonzero density</u>**

- ratios of moments of baryon number fluctuations
- baryon number fluctuations at freeze-out
- the radius of convergence

The Taylor expansion method

- Taylor-expansion of the pressure

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V,T,\mu_u,\mu_d,\mu_s) = \sum_{i,j,k} c_{i,j,k}^{u,d,s} \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_d}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k$$

- calculate Taylor coefficients at fixed temperature

- no sign-problem: all simulations at $\mu = 0$

$$c_{i,j,k}^{u,d,s} \equiv \frac{1}{i!j!k!} \frac{1}{VT^3} \cdot \left. \frac{\partial^i \partial^j \partial^k \ln Z}{\partial(\frac{\mu_u}{T})^i \partial(\frac{\mu_d}{T})^j \partial(\frac{\mu_s}{T})^k} \right|_{\mu_{u,d,s}=0}$$

- expansion coefficients reflect fluctuations of various quantum numbers

generalized susceptibilities

$$2! c_2^X = \chi_2^X = \frac{1}{VT^3} \left(\left\langle X^2 \right\rangle - \left\langle X \right\rangle^2 \right) \qquad \text{quadratic fluctuations}$$

$$4! c_4^X = \chi_4^X = \frac{1}{VT^3} \left(\left\langle X^4 \right\rangle - 3 \left\langle X^2 \right\rangle^2 \right) \qquad \text{quartic fluctuations}$$

$$X = u, d, s, B, Q, S, \cdots$$

The Taylor expansion method

- formulate all operators in terms of space-time, color (and spin) traces:

$$
\begin{aligned}
\frac{\partial(\ln\det M)}{\partial\mu} &= \mathcal{D}_1 = \mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}\right)\\
\frac{\partial^2(\ln\det M)}{\partial\mu^2} &= \mathcal{D}_2 = \mathrm{Tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) - \mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\right)\\
\frac{\partial^3(\ln\det M)}{\partial\mu^3} &= \mathcal{D}_3 = \mathrm{Tr}\left(M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) - 3\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right)\\
&\quad +2\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\right)\\
\frac{\partial^4(\ln\det M)}{\partial\mu^4} &= \mathcal{D}_4 = \mathrm{Tr}\left(M^{-1}\frac{\partial^4 M}{\partial\mu^4}\right) - 4\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right)\\
&\quad -3\mathrm{Tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) + 12\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right)\\
&\quad -6\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\right)
\end{aligned}
$$

- evaluate all traces by noisy estimators:

$$\mathrm{Tr}\left(\frac{\partial^{n_1}M}{\partial\mu^{n_1}}M^{-1}\frac{\partial^{n_2}M}{\partial\mu^{n_2}}\cdots M^{-1}\right) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\eta_k^\dagger\frac{\partial^{n_1}M}{\partial\mu^{n_1}}M^{-1}\frac{\partial^{n_2}M}{\partial\mu^{n_2}}\cdots M^{-1}\eta_k$$

with N random vectors, satisfying $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\eta_{n,i}^*\eta_{n,j} = \delta_{i,j}$

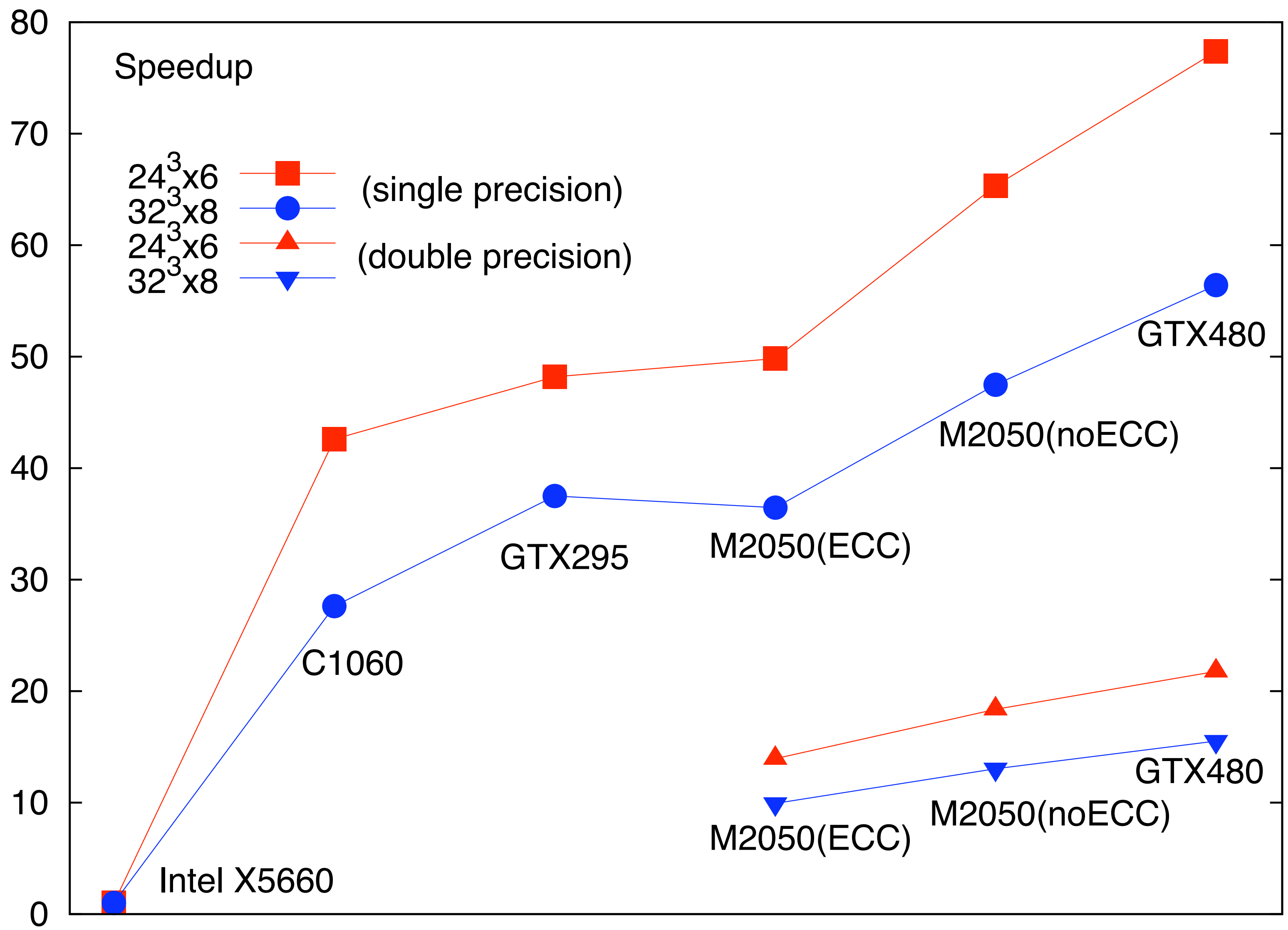
- construct expansion coefficients from $\mathcal{D}_n^u, \mathcal{D}_n^d, \mathcal{D}_n^s,$ with unbiased estimators

$$c_{2,0,0}^{u,d,s} = \frac{1}{2}\frac{N_\tau}{N_\sigma^3}\left(\langle\mathcal{D}_2^u\rangle + \left\langle\left(\mathcal{D}_1^u\right)^2\right\rangle\right)$$

Allton et al., Phys.Rev.D66:074507,2002.

Lattice QCD on GPU's

- use effective implementation of the CG-solver for p4-improved staggered fermions on GPU's

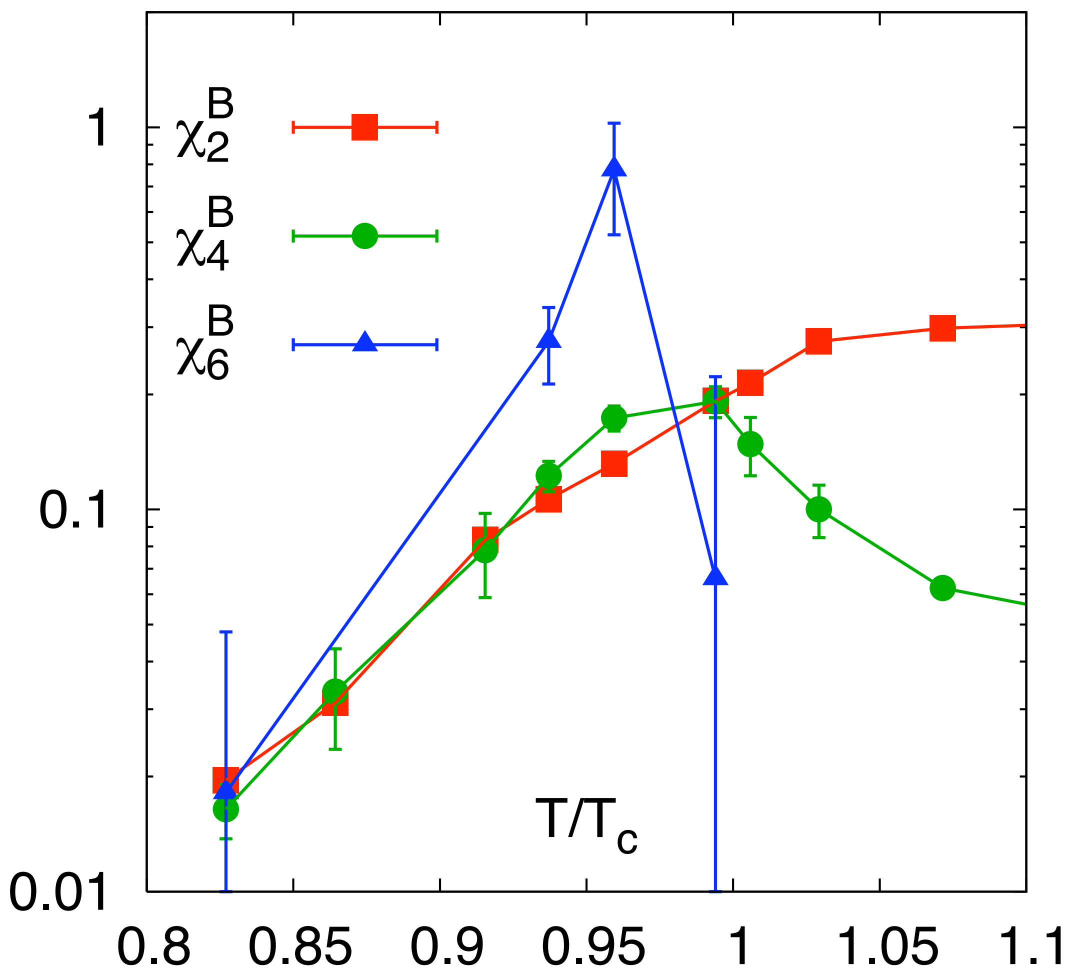


courtesy O. Kaczmarek

- use CUDA by NVIDIA
- most important optimization: obtain coalesced memory access by a spacial storage pattern of the fields (color index is the most outer index)
- see also:
  - Egri et al., Comput.Phys.Commun. 177:631-639,2007.
  - Clark, PoS LAT2009:003,2009.
  - Bonati et al., PoS(Lattice 2010)324

Baryon number fluctuations

$16^3 \times 4,\ m_q = m_s/10$



Analyzing the critical behavior:

scaling field (chiral limit):

$$t = \frac{1}{t_0}\left(\frac{T-T_c}{T_c} + \kappa\left(\frac{\mu_B}{T}\right)^2\right)$$

free energy:

$$f = A_\pm |t|^{2-\alpha} + \text{regular}$$

critical exponent:

$$-0.15 < \alpha < -0.11$$

$$\chi_2^B \sim \mp 2A_\pm(2-\alpha)\kappa\,|t|^{1-\alpha} + \text{regular}$$

$$\chi_4^B \sim -12A_\pm(2-\alpha)(1-\alpha)\kappa^2\,|t|^{-\alpha} + \text{regular}$$ ⟶ kink (chiral limit)

$$\chi_6^B \sim \mp 120A_\pm(2-\alpha)(1-\alpha)(-\alpha)\kappa^3\,|t|^{-1-\alpha} + \text{regular}$$ ⟶ divergent (chiral limit)

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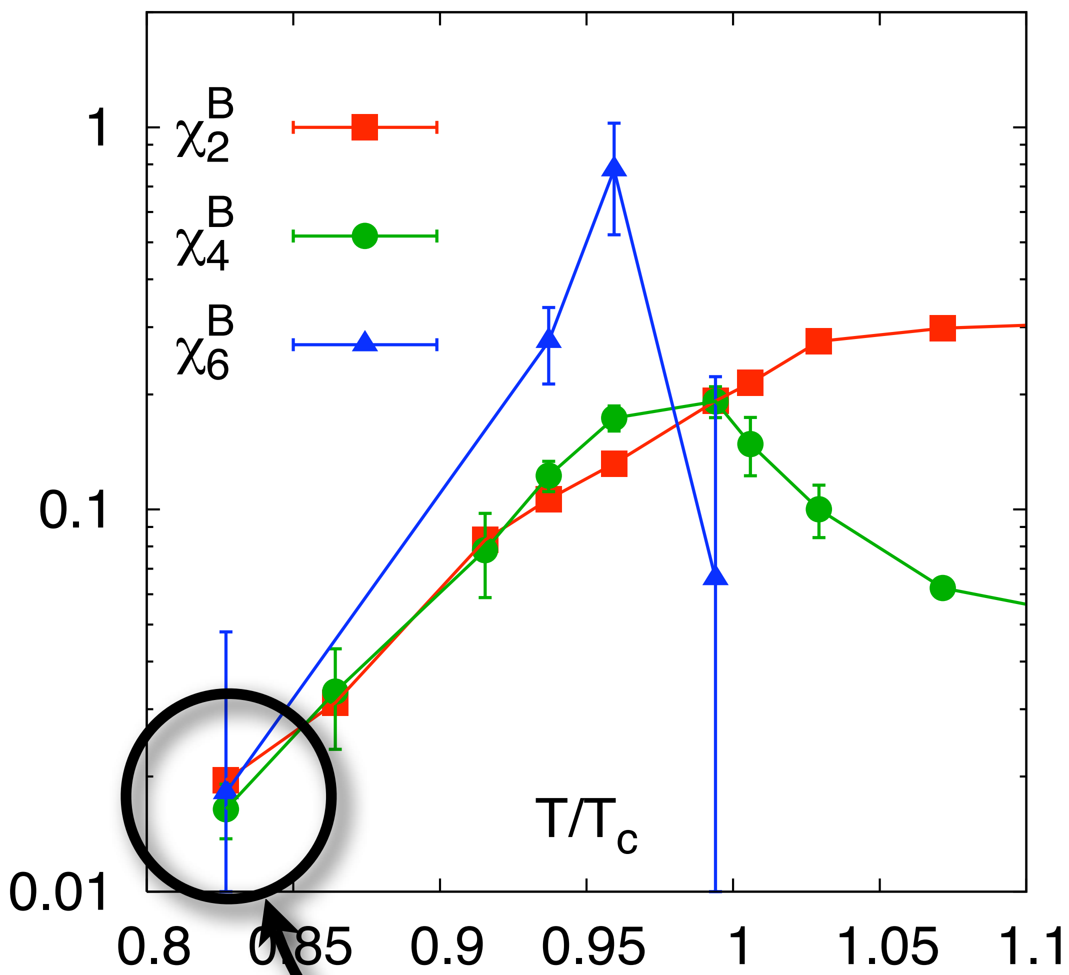

GPU point, 4500 independent configuratons, 1800 random vectors each

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Baryon number fluctuations

hadron resonance gas

$$\ln Z(T,V,\mu_B,\mu_S,\mu_Q) = \sum_{i\in hadrons} \ln Z_{m_i}(T,V,\mu_B,\mu_S,\mu_Q)$$

$$\sum_{i\in mesons} \ln Z^B_{m_i}(T,V,\mu_S,\mu_Q) + \sum_{i\in baryons} \ln Z^F_{m_i}(T,V,\mu_B,\mu_S,\mu_Q)$$

mesons:

$$\frac{p_i}{T^4} = \frac{d_i}{\pi^2}\left(\frac{m_i}{T}\right)^2 \sum_{l=1}^{\infty}(+1)^{l+1} l^{-2} K_2(lm_i/T)\cosh(lS_i\mu_S/T + lQ_i\mu_Q/T)$$

baryons:

$$\frac{p_i}{T^4} = \frac{d_i}{\pi^2}\left(\frac{m_i}{T}\right)^2 \sum_{l=1}^{\infty}(-1)^{l+1} l^{-2} K_2(lm_i/T)\cosh(lB_i\mu_B/T + lS_i\mu_S/T + lQ_i\mu_Q/T)$$

Boltzmann approximation

ratios are independent of spectrum and volume

$\longrightarrow$ possibly large parts of cut-off effects cancel

3 ratios:

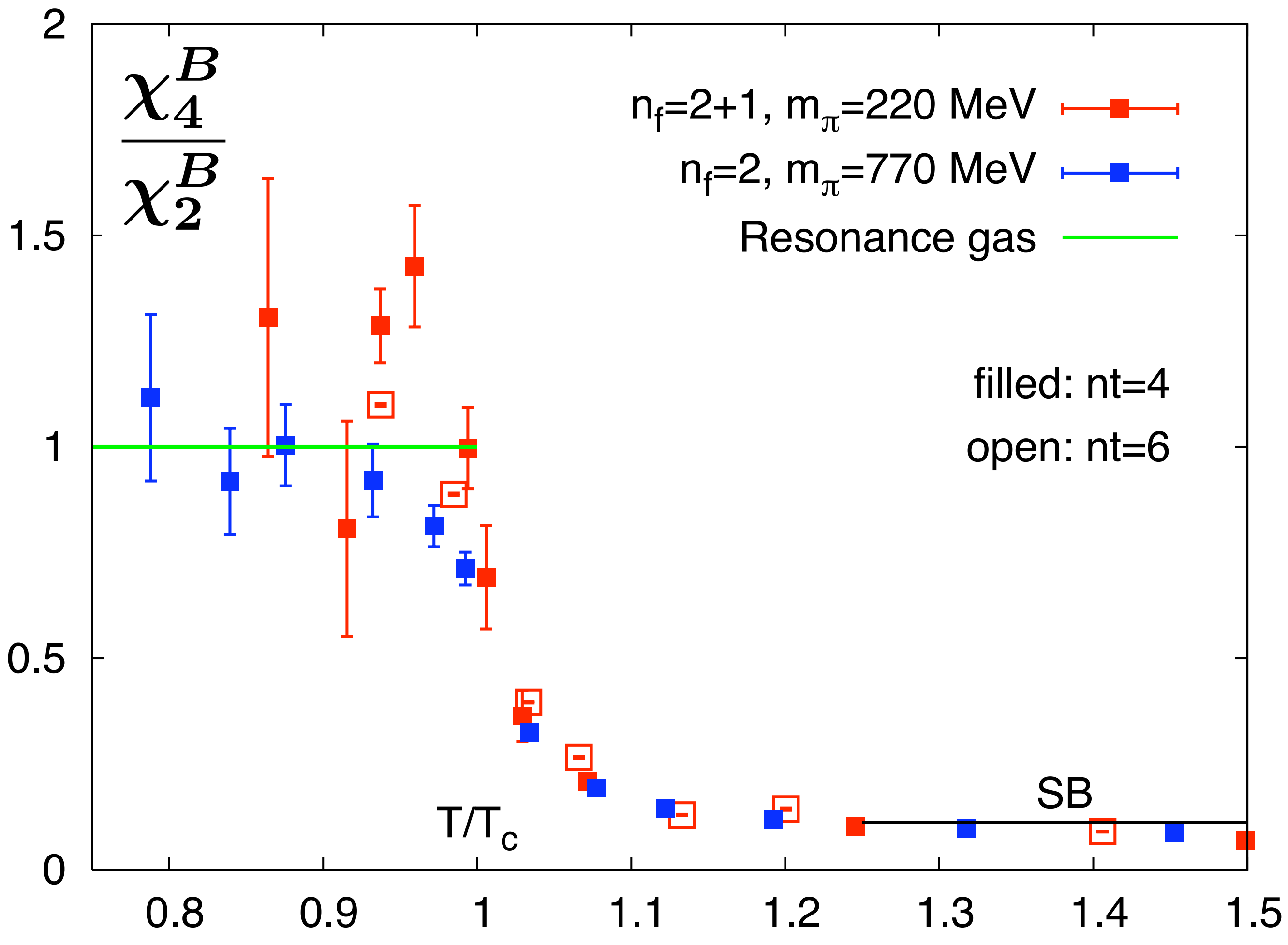
$$\frac{\chi_4^B}{\chi_2^B} = \kappa\sigma^2 = \frac{B^4}{B^2} = 1$$

$$\frac{\chi_3^B}{\chi_2^B} = S\sigma = \frac{B^3}{B^2}\tanh(\mu_B/T)$$

$$\frac{\chi_2^B}{\chi_1^B} = \sigma^2/N_B = \frac{B^2}{B^1}\coth(\mu_B/T)$$

Baryon number fluctuations

- Kurtosis times variance

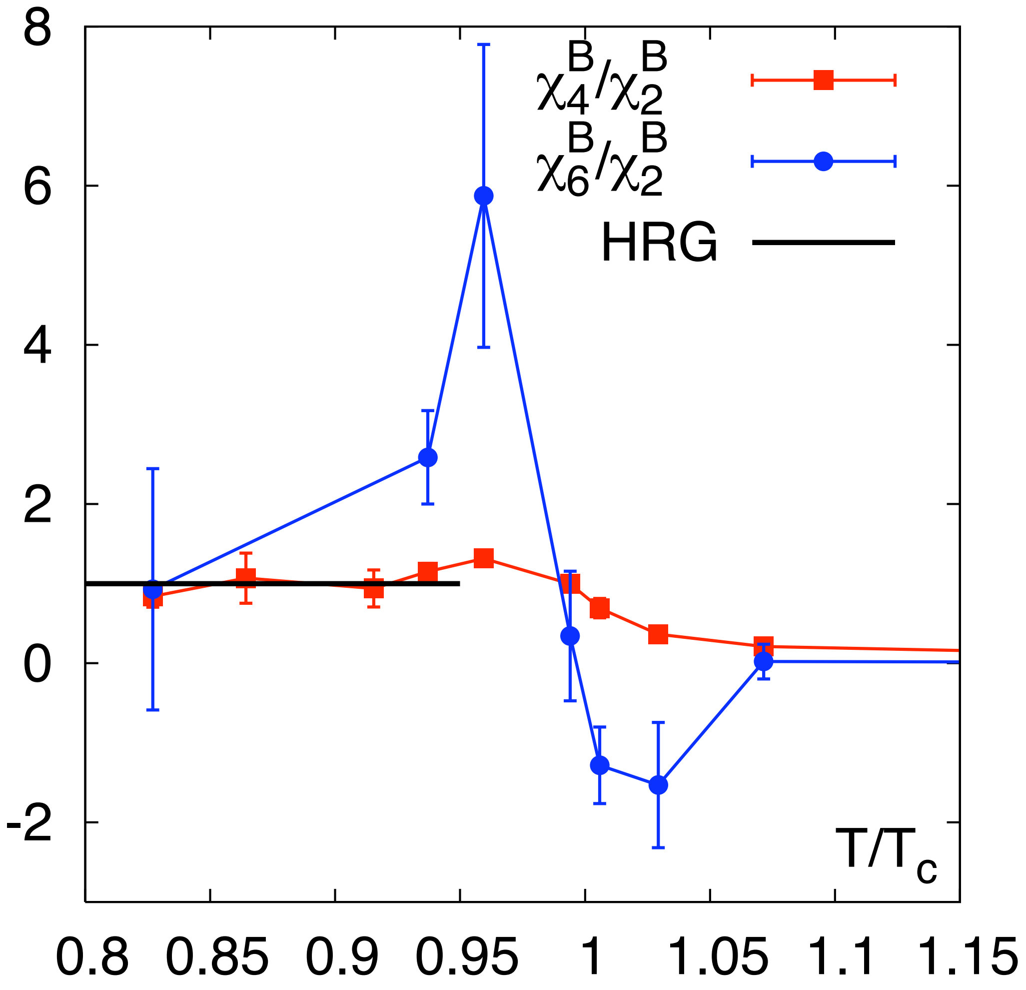


red: CS, J.Phys.G35 (2008) 104093.
blue: Allton et all., Phys. Rev. D71 (2005) 054508.

- sensitive to relevant quantum numbers in the medium
- divergent at the critical point (Z(2) Universality class: $\alpha > 0$)

Baryon number fluctuations

- sixth order fluctuations



[CS, arXiv:1007.5164]

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Baryon number fluctuations

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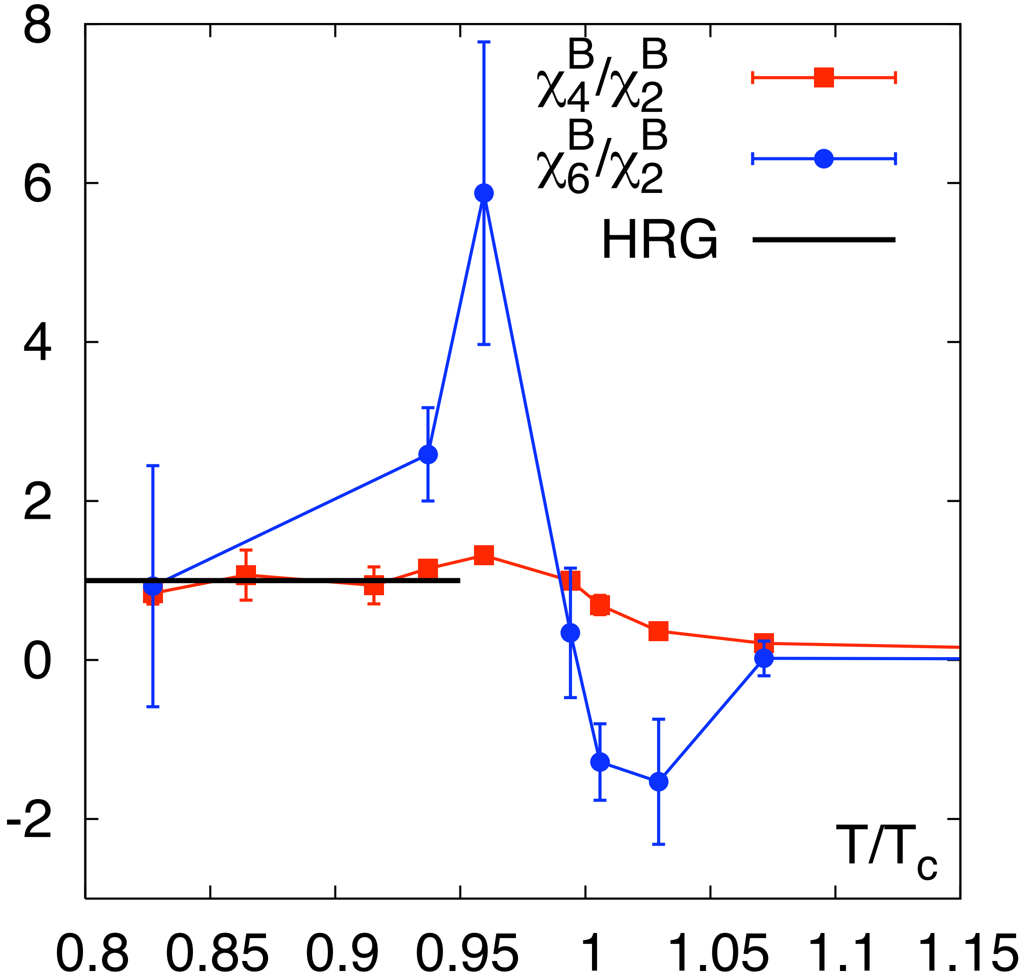

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- sensitive to relevant quantum numbers in the medium
- divergent at the critical point

Use parametrization of Freeze-out curve to connect to STAR measurements of net-proton number

$$T(\mu_B) = 0.166\ \mathrm{GeV} - 0.139\ \mathrm{GeV}^{-1}\mu_B^2 - 0.053\ \mathrm{GeV}^{-3}\mu_B^4$$

$$\mu_B(\sqrt{s}) = \frac{1.308\ \mathrm{GeV}}{1 + 0.273\ GeV^{-1}\sqrt{s}}$$

[Cleymans et al., Phys. Rev. C 63 (2006) 034905]

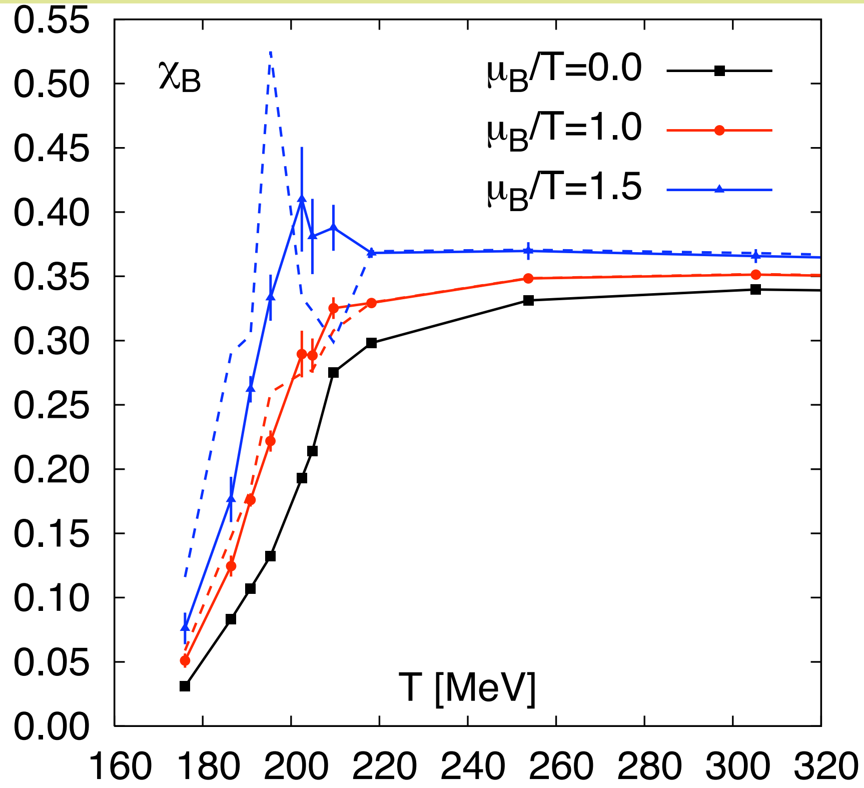
Hadronic fluctuations

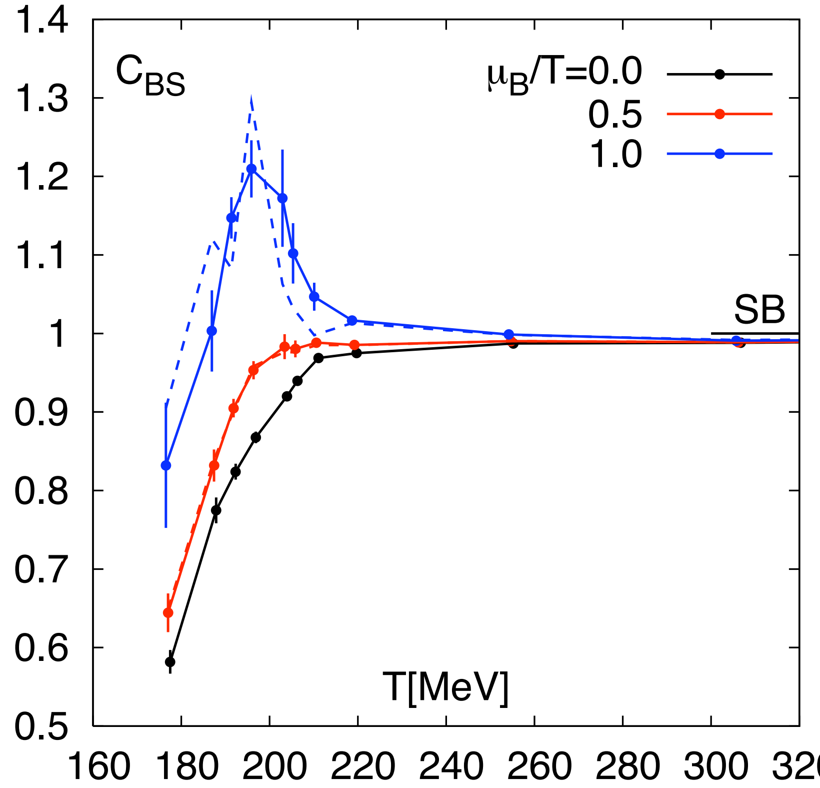
at $\mu_B > 0 \ (\mu_S = \mu_Q = 0)$

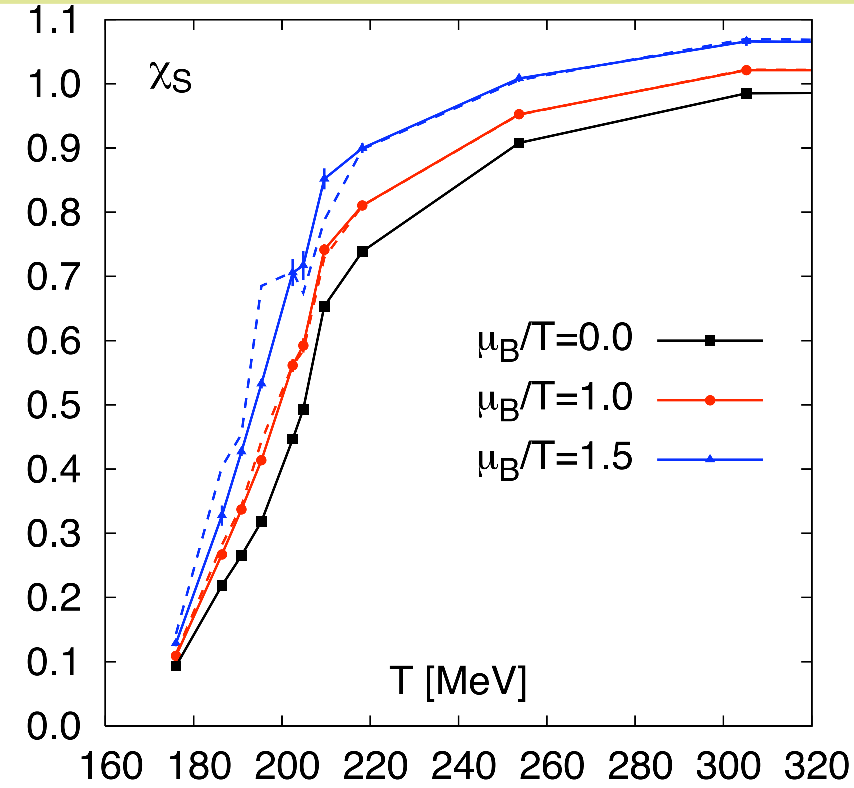
baryon number fluctuations	baryon number - strangeness correlations	strangeness fluctuations

$$\chi_B = 2c_2^B + 12c_4^B \left(\frac{\mu_B}{T}\right)^2 + \cdots$$

$$\chi_S = 2c_{0,2}^{B,S} + 2c_{2,2}^{B,S} \left(\frac{\mu_B}{T}\right)^2 + \cdots$$







$$C_{BS} = \frac{c_{1,1}^{B,S} + 3c_{3,1}^{B,S}\left(\frac{\mu_B}{T}\right)^2 + \cdots}{\chi_S\left(\frac{\mu_B}{T}\right)}$$

$\longrightarrow$ LO introduces a peak in the fluctuations/correlations, NLO shifts the peak towards smaller temperatures

$\longrightarrow$ truncation errors become large at $\mu_B/T \gtrsim 1.5$

Hadronic fluctuations

at $\mu_B > 0$ $(\mu_S = \mu_Q = 0)$

Taylor expansion of ratios:

$$\frac{\chi_2}{\chi_1} = \left(\frac{\mu}{T}\right)^{-1} + \frac{4c_4}{c_2}\left(\frac{\mu}{T}\right)^{1} + \left(-\frac{8c_4^2}{c_2^2} + \frac{12c_6}{c_2}\right)\left(\frac{\mu}{T}\right)^{3} + \mathcal{O}\left[\left(\frac{\mu}{T}\right)^{5}\right]$$

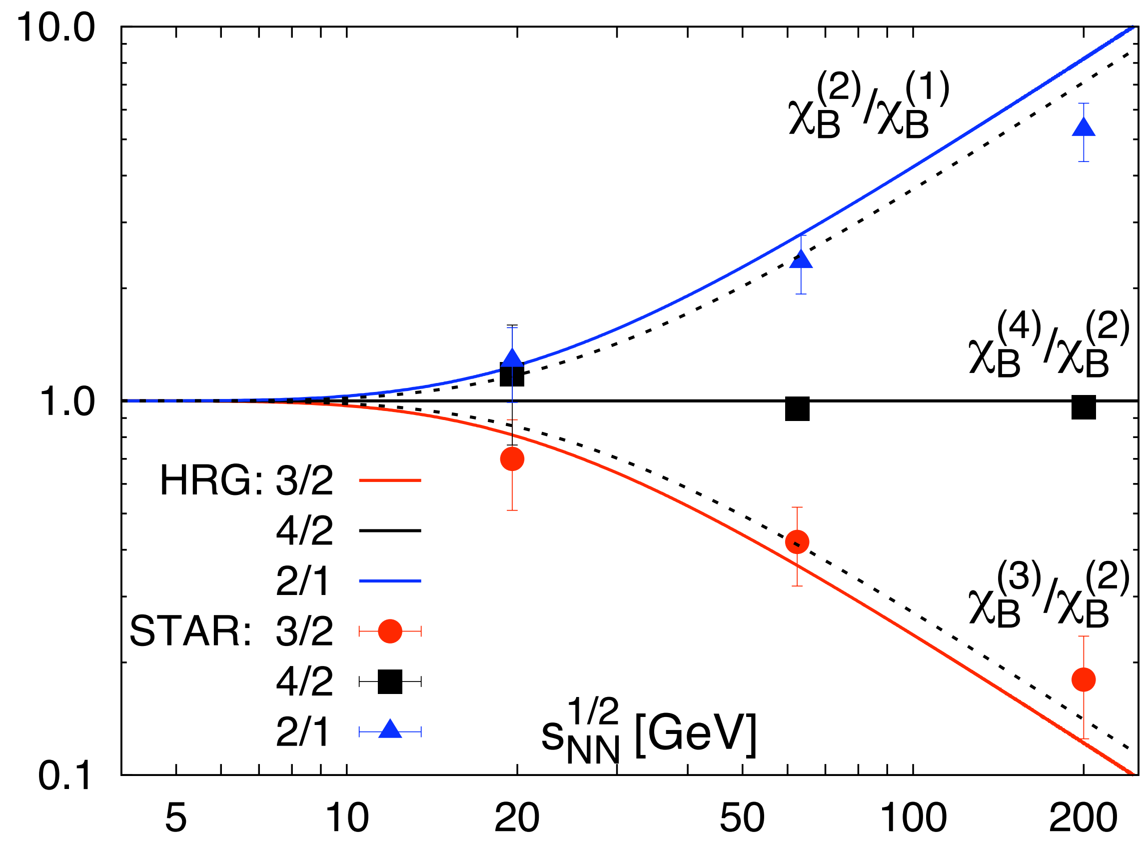
$$\frac{\chi_3}{\chi_2} = \frac{12c_4}{c_2}\left(\frac{\mu}{T}\right)^{1} + \left(-\frac{72c_4^2}{c_2^2} + \frac{60c_6}{c_2}\right)\left(\frac{\mu}{T}\right)^{3} + \mathcal{O}\left[\left(\frac{\mu}{T}\right)^{5}\right]$$

$$\frac{\chi_4}{\chi_2} = \frac{12c_4}{c_2} + \left(-\frac{72c_4^2}{c_2^2} + \frac{180c_6}{c_2}\right)\left(\frac{\mu}{T}\right)^{2} + \mathcal{O}\left[\left(\frac{\mu}{T}\right)^{4}\right]$$

- use parametrization of the freeze-out line to obtain μ/T as function of $\sqrt{s}$
- use multi-histogram re-weighting to interpolate between temperatures
- caution: neglecting μ_Q, μ_S dependence

Baryonic fluctuations at freeze-out

HRG vs. Experiment:



[Karsch, Redlich, arXiv:1007.2581]

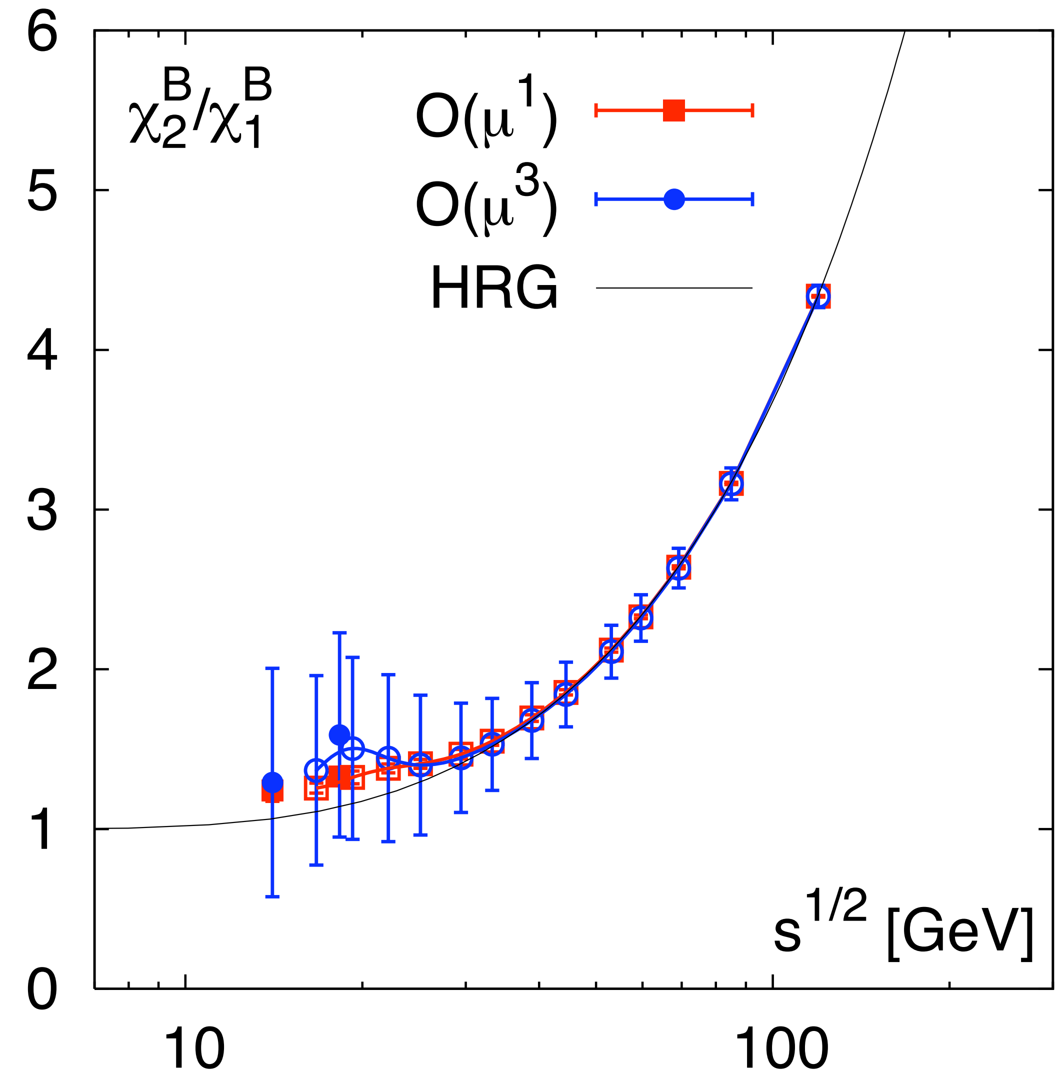
[STAR data: Aggarwal et al, PRL (2010) 022302]

- net-proton number fluctuations can be described by the HRG

  solid lines: $\mu_Q \neq 0, \mu_S \neq 0$

  dashed lines: $\mu_Q = 0, \mu_S = 0$

Lattice vs. HRG:



[CS, arXiv:1007.5164]

- fluctuations increase for small $\sqrt{s}$
- sensitive to truncation of the series due to close radius of convergence

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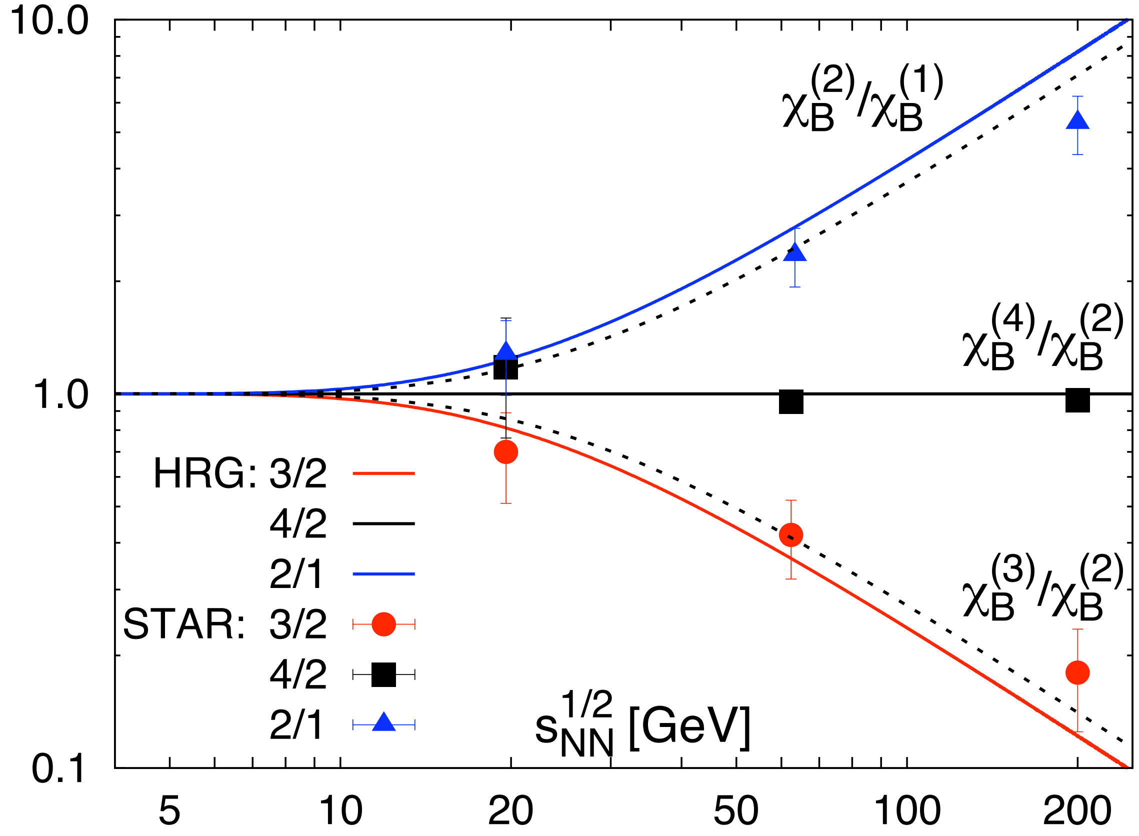

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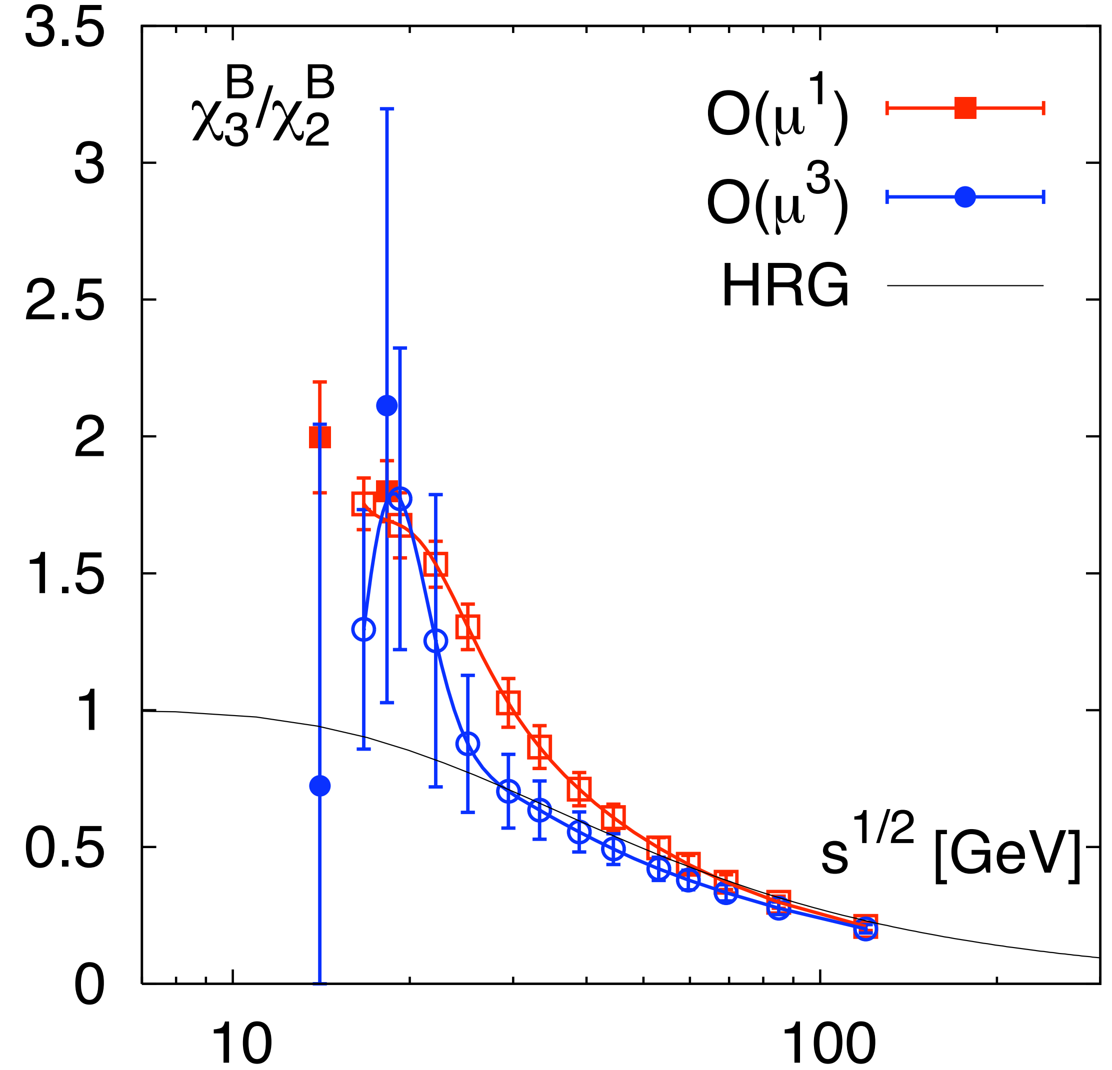

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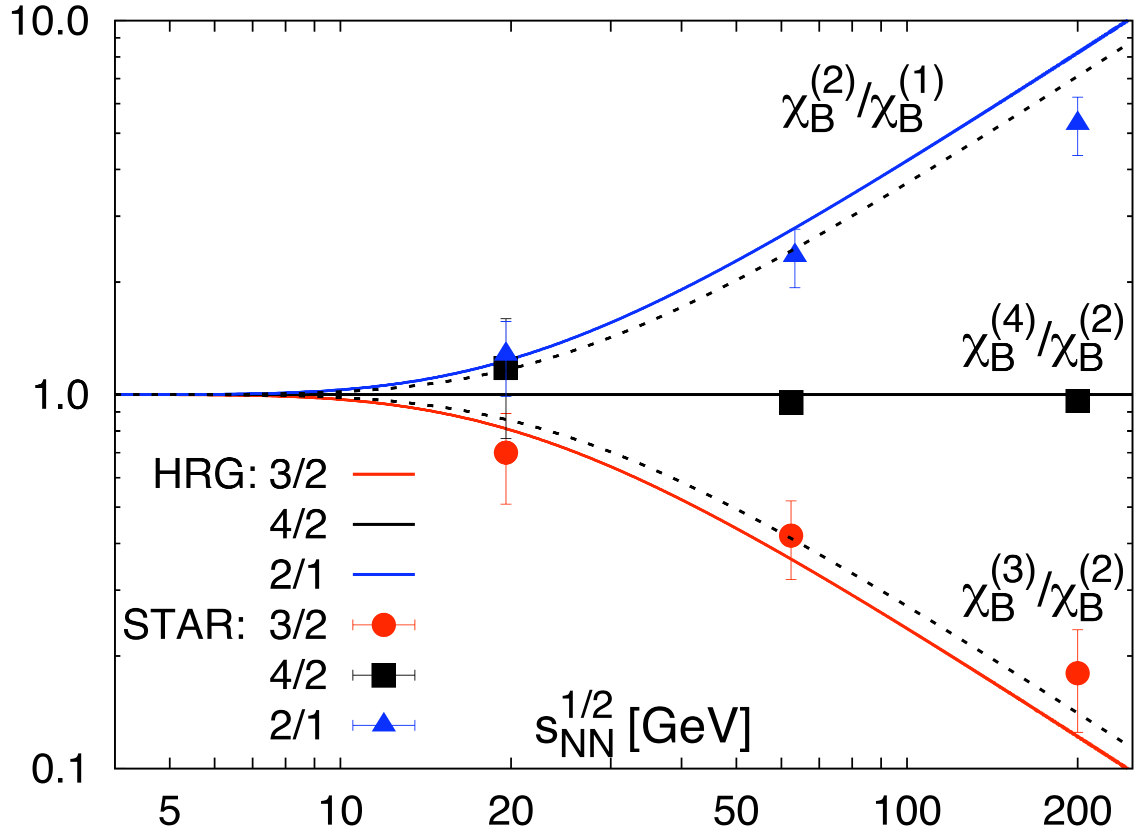

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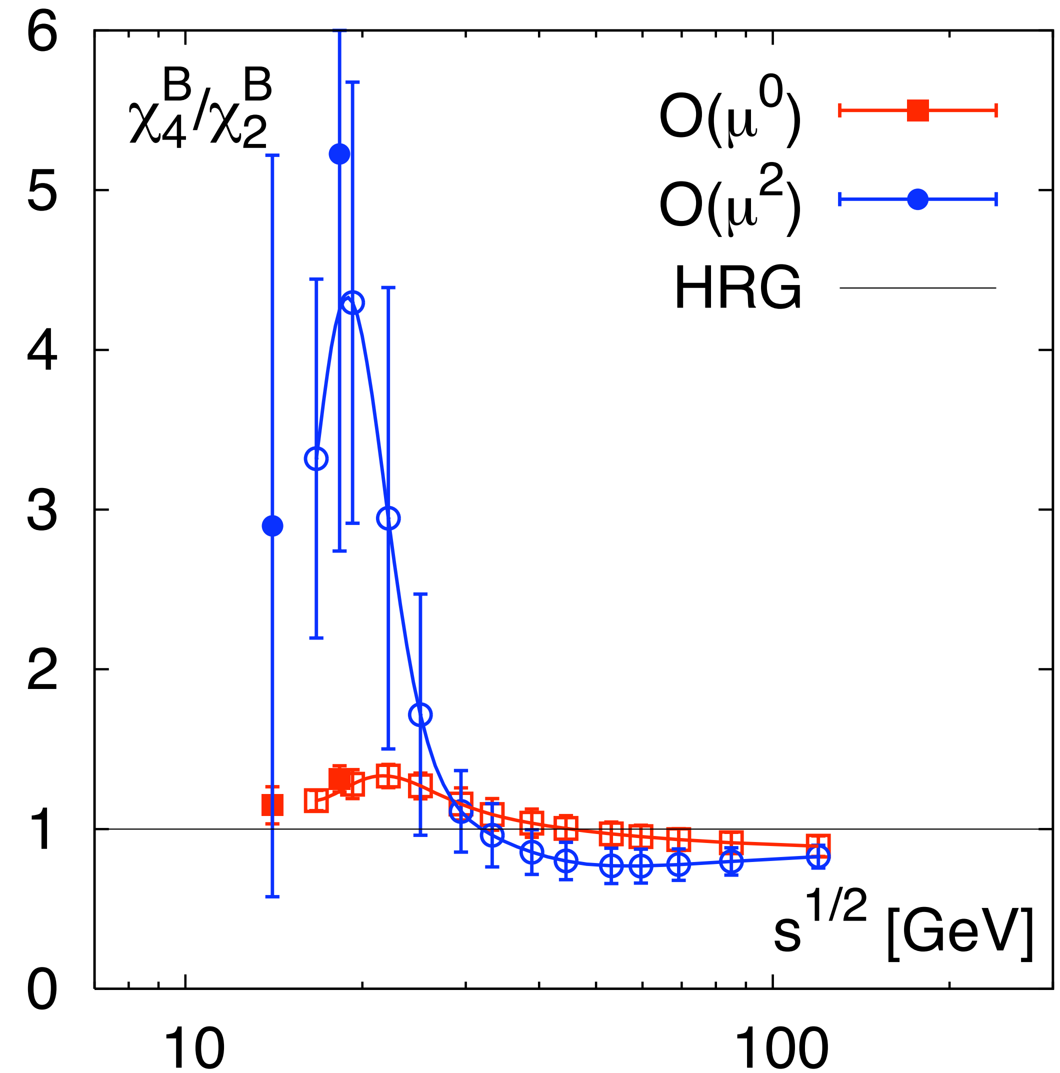

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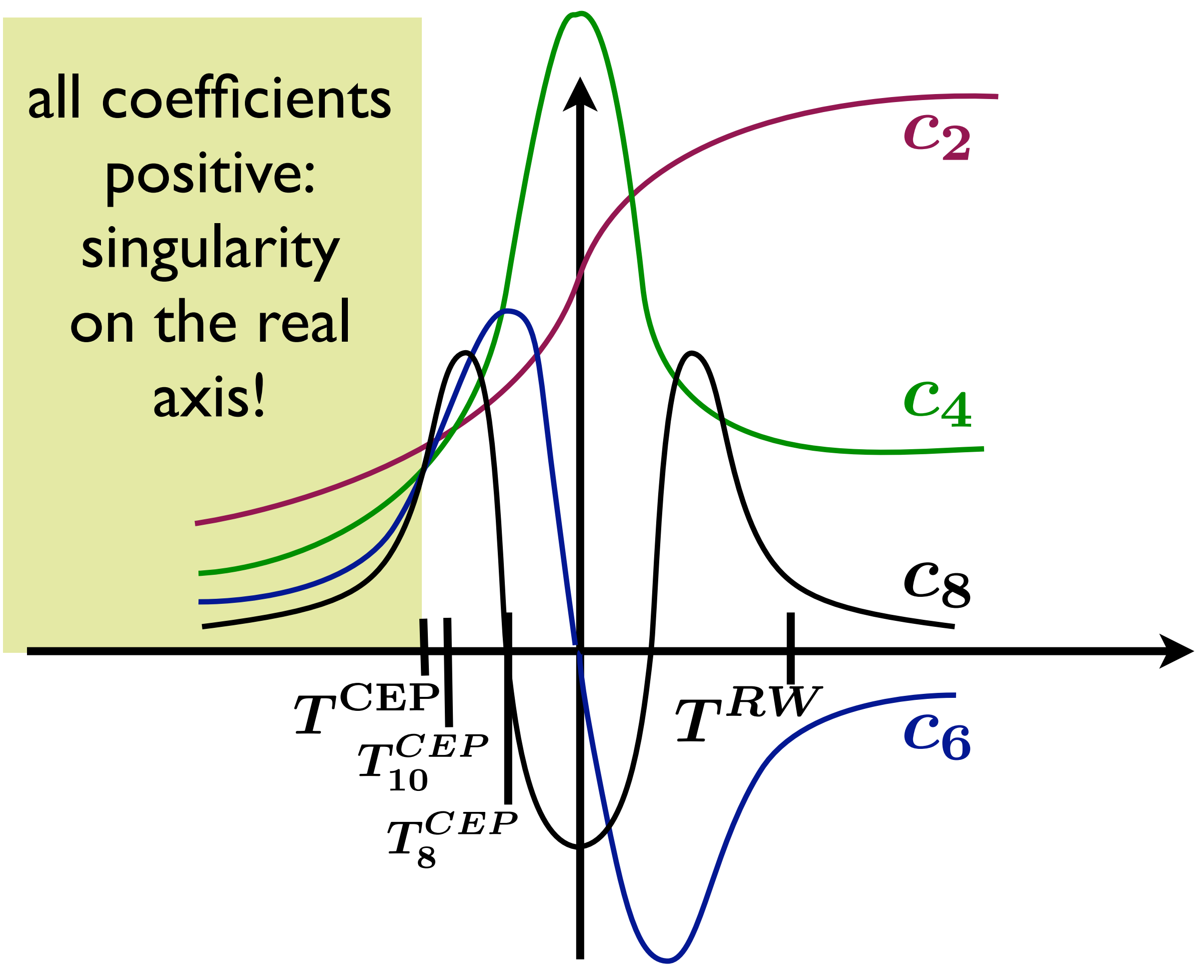
The critical endpoint

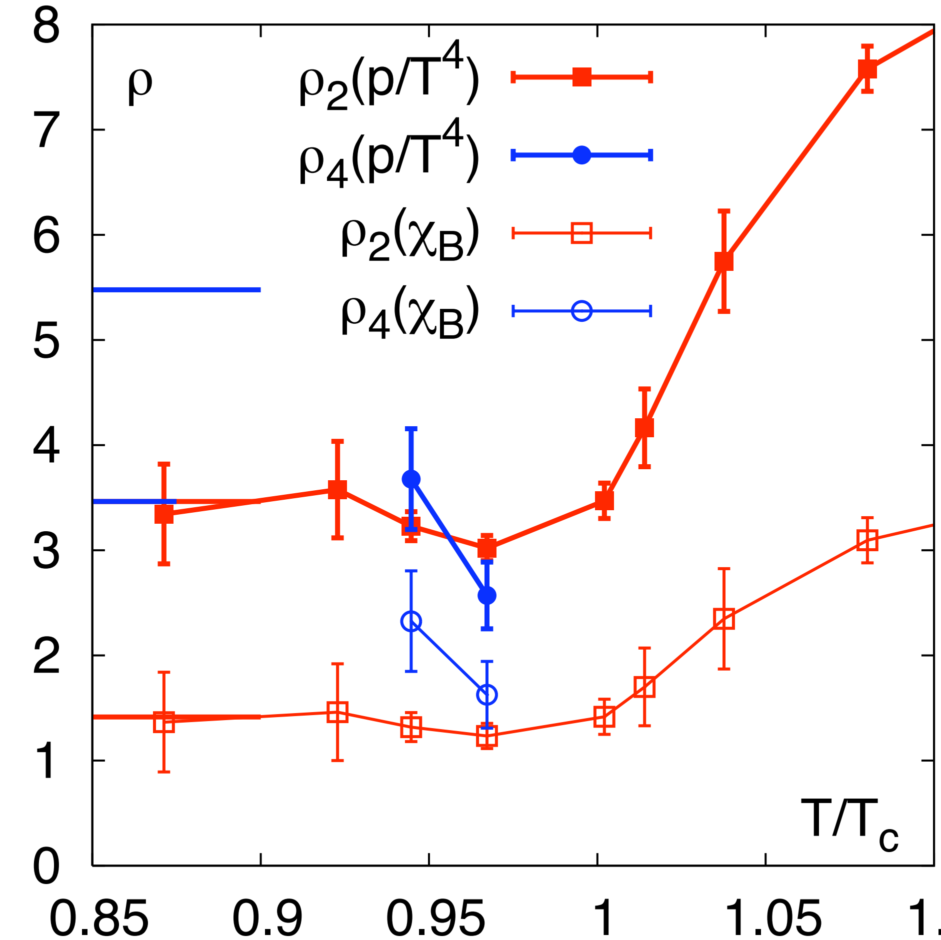
<u>method for locating of the CEP:</u>

- determine largest temperature where all coefficients are positive $\rightarrow T^{\mathrm{CEP}}$
- determine the radius of convergence at this temperature $\rightarrow \mu^{\mathrm{CEP}}$

$$p = c_0 + c_2 \left(\mu_B/T\right)^2 + c_4 \left(\mu_B/T\right)^4 + \cdots$$

$$\chi_B = 2c_2 + 12c_4 \left(\mu_B/T\right)^2 + 30c_6 \left(\mu_B/T\right)^4 + \cdots$$





[CS, arXiv:1007.5164]

first non-trivial estimate of T^{CEP} by c_8
second non-trivial estimate of T^{CEP} by c_{10}

$$\rho_n(p) = \sqrt{c_n/c_{n+2}}$$

$$\rho = \lim_{n\rightarrow\infty} \rho_n$$

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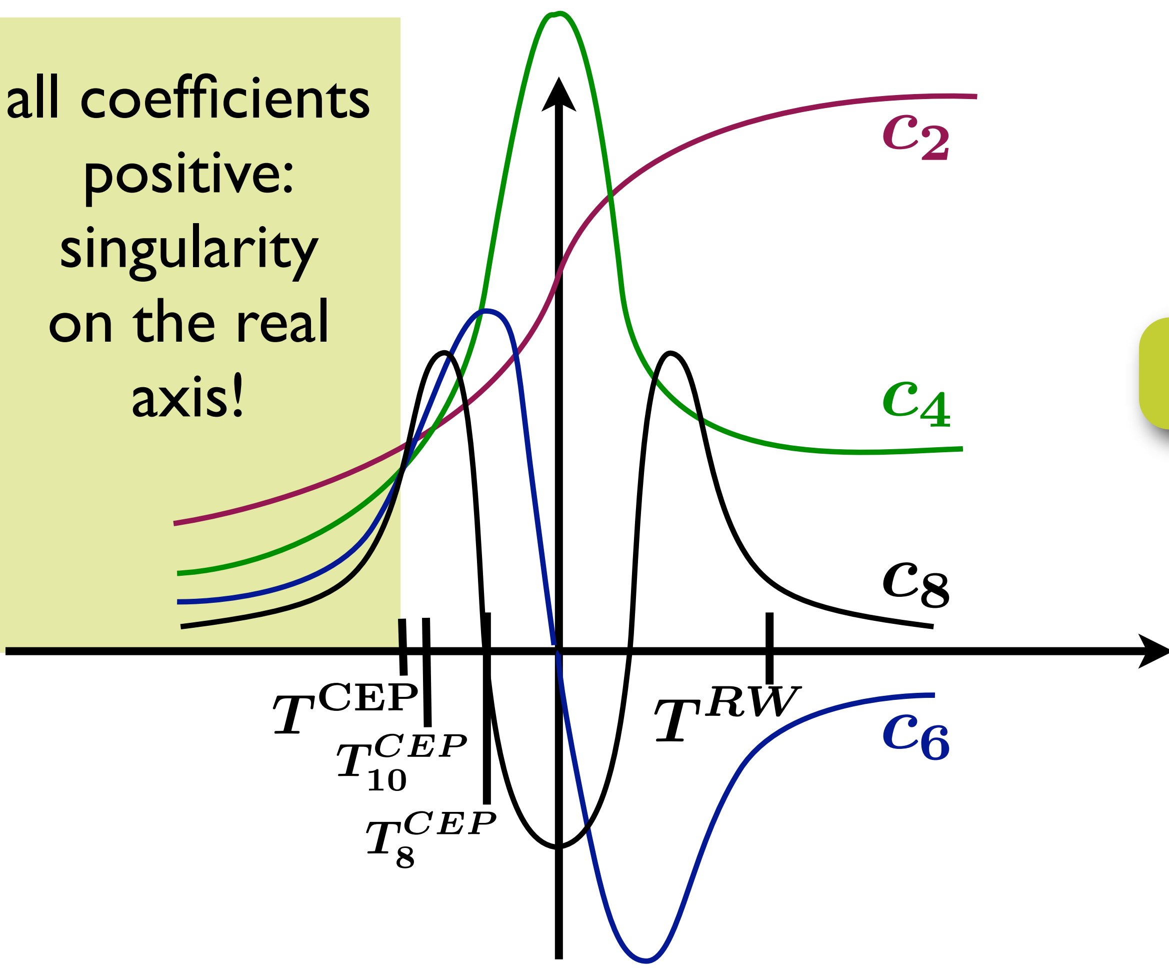
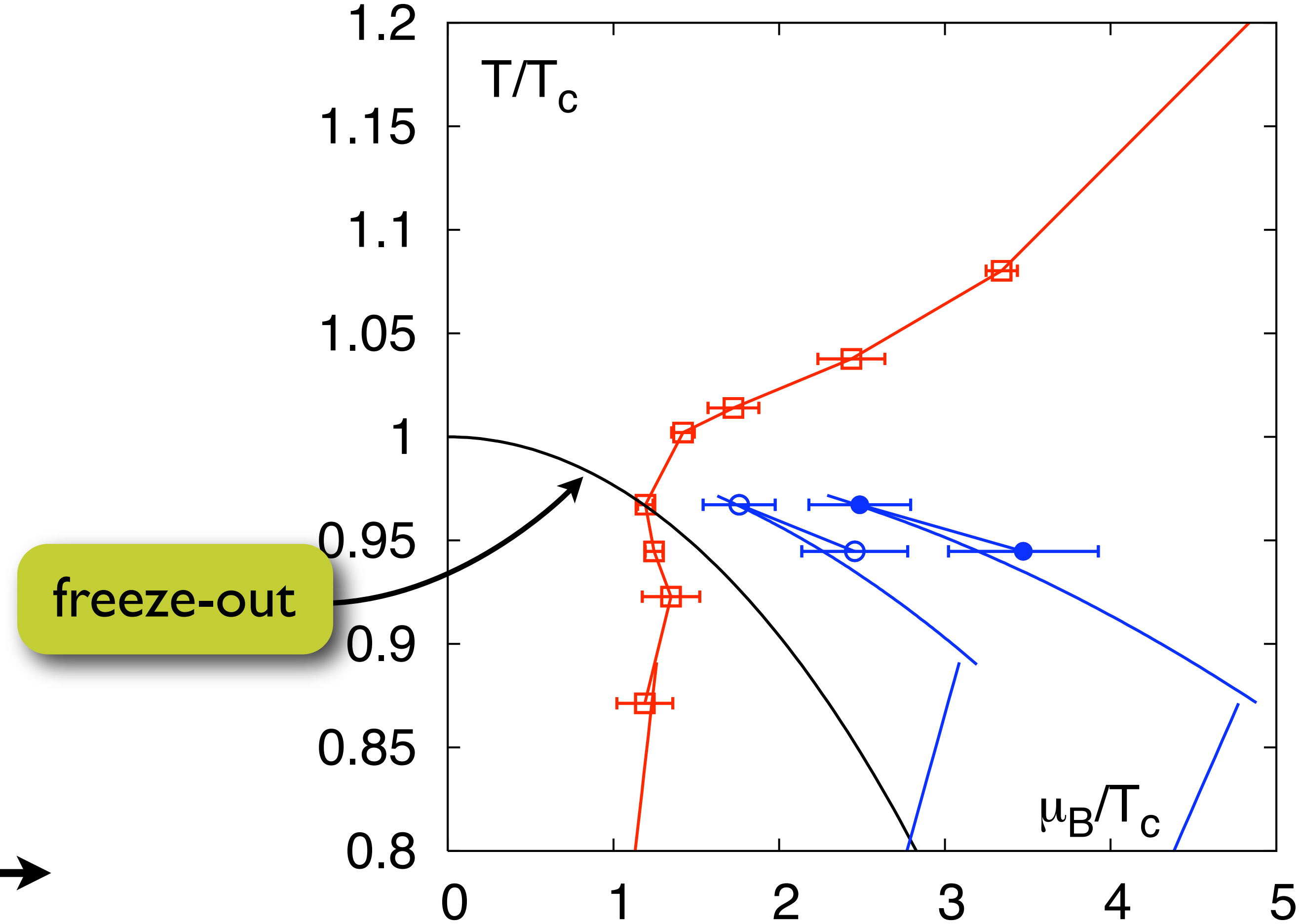
all coefficients positive: singularity on the real axis!





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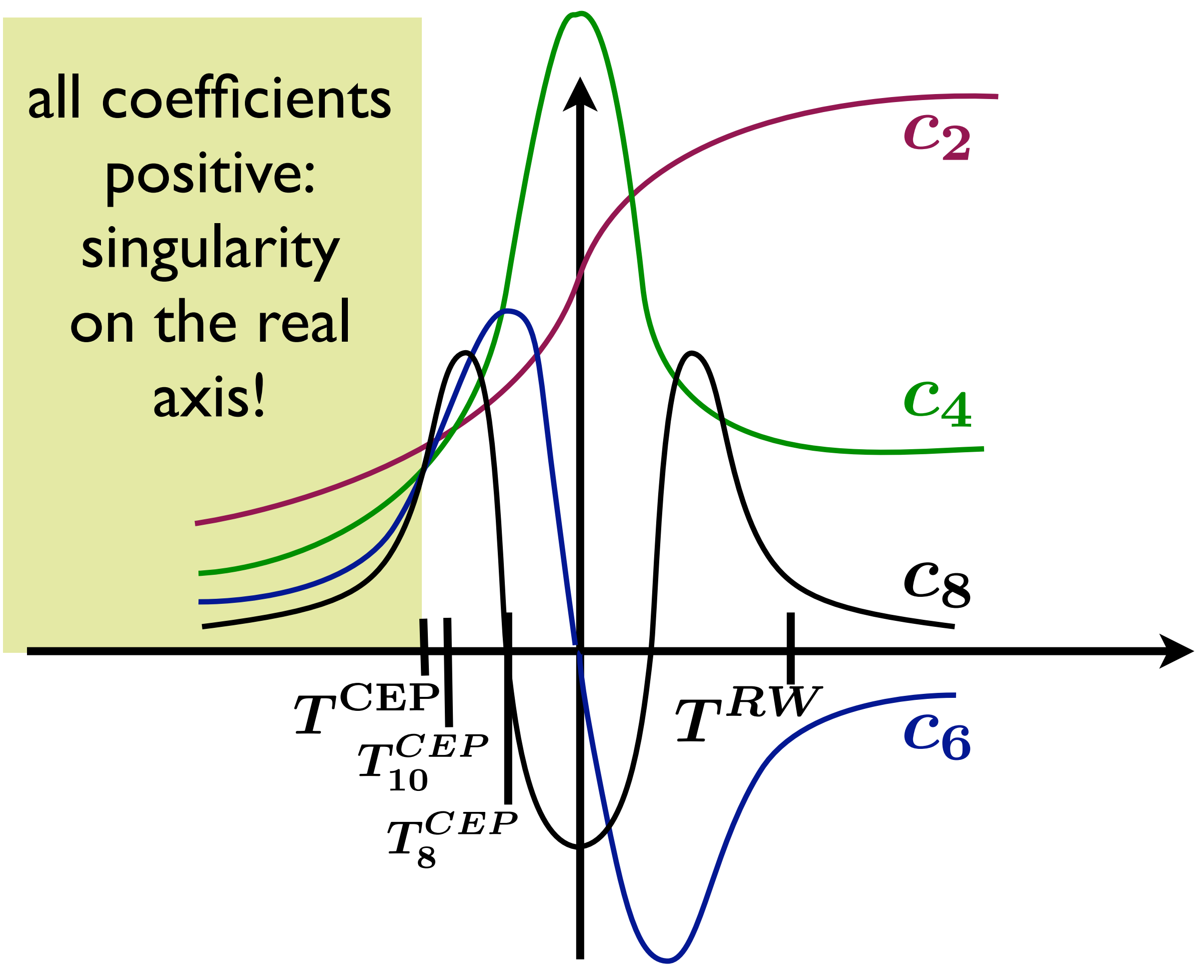

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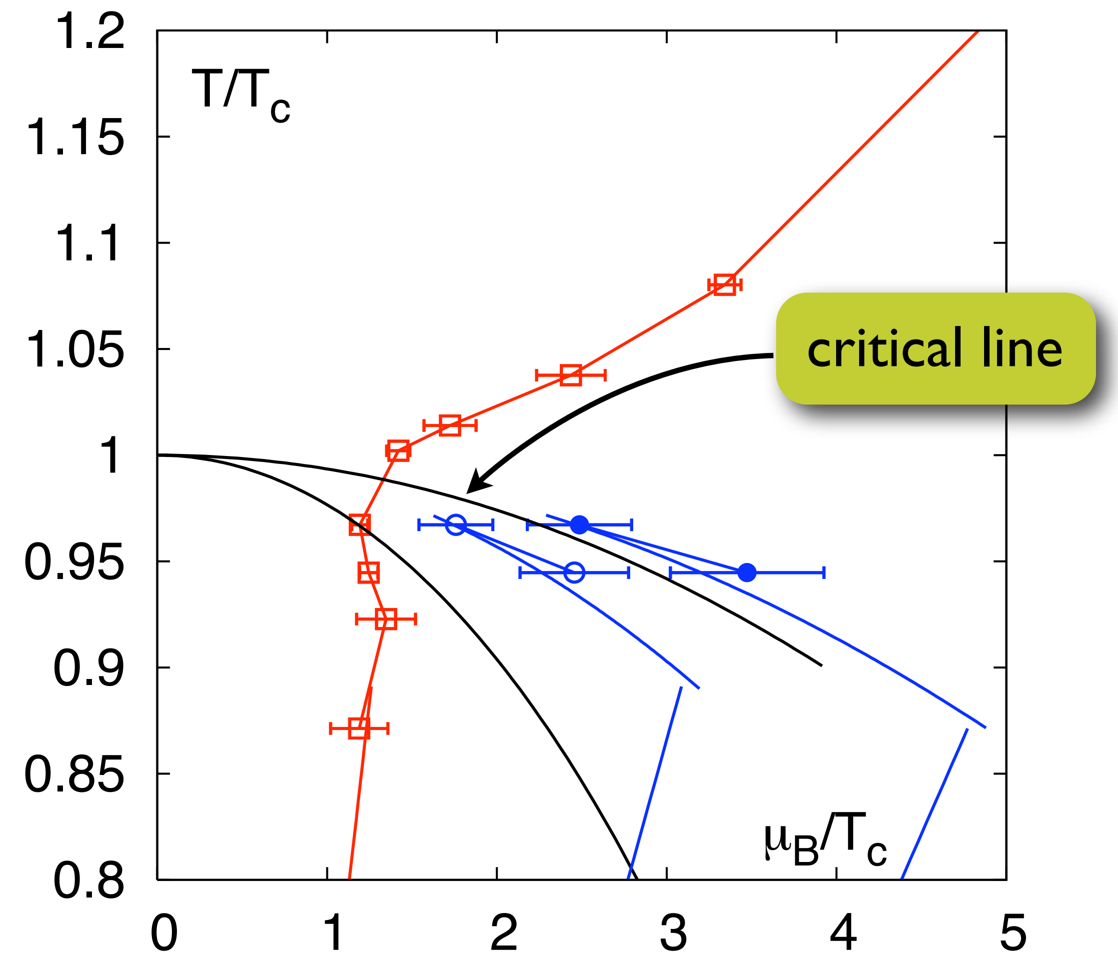

[CS, arXiv:1007.5164]

- radius of convergence is consistent with critical line in the chiral limit (for determination of critical line see talk by F. Karsch)

Summary

- We have calculated higher order baryonic fluctuations, which are more and more sensitive to the critical behavior of QCD (in the chrial limit and close to the critical point

- Ratios of baryonic susceptibilities are sensitive to the relevant degrees of freedom but are rather independent on the spectrum and volume. For Low temperatures the lattice data can be explained by the HRG model.

→ results need to be confirmed in the continuum limit

- STAR data on net-proton fluctuations can be described the lattice data (and the HRG model) for $\sqrt{s} \gtrsim (10-20)\ \mathrm{GeV}$

- Estimates of the radius of convergence are consistent with the critical line in the chiral limit as determined from a fit of a mixed chiral and light quark susceptibility to the corresponding universal scaling function (see talk by F. Karsch).

- 8th order fluctuations are necessary to obtain a first nontrivial estimate of the critical point.