Heavy-light meson properties from non-perturbative HQET

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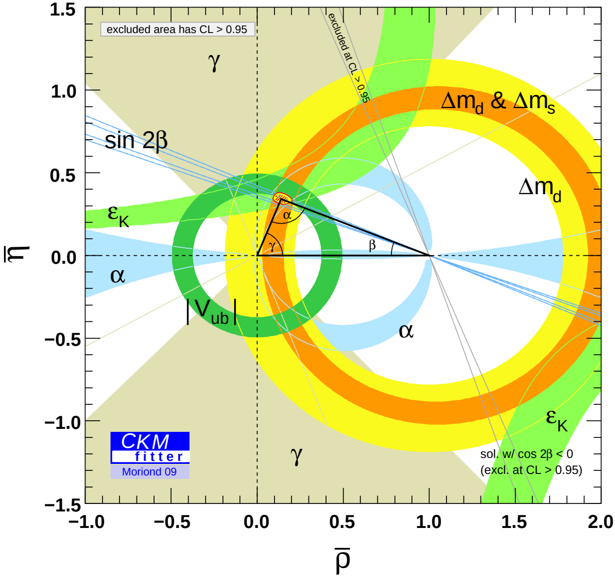
Introduction

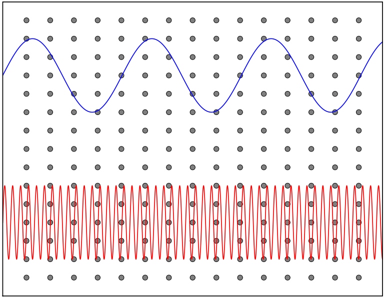
- Barring a direct discovery of BSM particles at the LHC, flavour physics is our best chance to discover new physics
- Precise determinations of CKM matrix elements depend on lattice QCD predictions
- Some tension reported in B_s and D_s decays
- Maybe one can overconstrain CKM unitarity triangle $\rightsquigarrow$ new physics


excluded area has CL > 0.95
excluded at CL > 0.95
γ
Δm_d & Δm_s
sin 2β
Δm_d
ε_K
α
β
γ
α
$|V_{ub}|$
α
ε_K
CKM fitter
Moriond 09
γ
sol. w/ cos 2β < 0
(excl. at CL > 0.95)
$\bar{\eta}$
$\bar{\rho}$
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

- Orbitally excited states of B mesons have recently been observed: B_{s1}, B^*_{s2}, B_1, B^*_2 (D0 2007, CDF 2007)
- Predictions of excitation energies from QCD highly desirable
- Other properties of excited mesons may also be accessible at least in theory
- Also desirable to understand $B - B^*$ spin splitting: why $m^2_{B^*} - m^2_B \approx m^2_{D^*} - m^2_D$?
- Spin splitting comes out too small in quenched approximation: quenching effect?

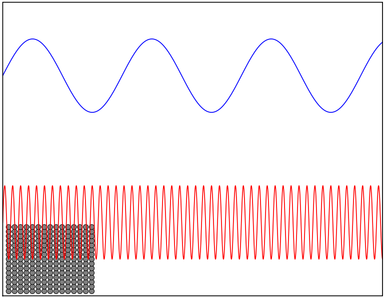
Problems for heavy quarks on the lattice

- Multi-scale problem: Need $L \gg m_\pi^{-1} \gg m_Q^{-1} \gg a$
- $a \gtrsim m_Q^{-1}$: large discretisation effects



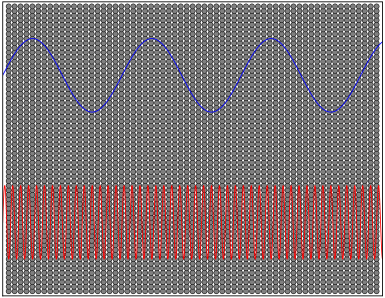
Problems for heavy quarks on the lattice

- Multi-scale problem: Need $L \gg m_\pi^{-1} \gg m_Q^{-1} \gg a$
- $L \lesssim m_\pi^{-1}$: large finite-volume effects



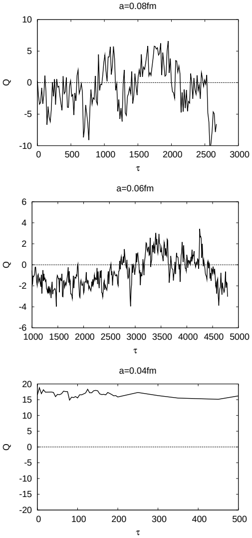
Problems for heavy quarks on the lattice

- Multi-scale problem: Need $L \gg m_\pi^{-1} \gg m_Q^{-1} \gg a$
- Large L/a leads to big computational effort



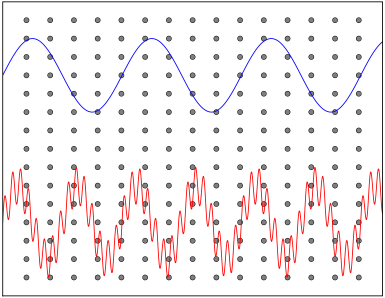
Critical slowing-down of HMC

- Question mark over fine lattice results . . .
- Autocorrelations of $Q_{\rm top}$ increase dramatically as $a \to 0$ [Schaefer, Sommer, Virotta, 2010]
- May not affect charm observables too much, but . . .
- $a \lesssim 0.04$ fm ensembles are not sampled correctly
- Can be understood in terms of the geometric construction using Wilson flow [Lüscher 2010]



Problems for heavy quarks on the lattice

- Multi-scale problem: Need $L \gg m_\pi^{-1} \gg m_Q^{-1} \gg a$
- Another way: effective theories – HQET, (m)NRQCD



HQET on the lattice

- Non-relativistic expansion of QCD Lagrangian:

$$\mathcal{L} = \overline{\psi}\left(D_0 - \frac{\mathbf{D}^2}{2M} - \frac{\sigma\cdot\mathbf{B}}{2M} + \dots\right)\psi$$

- Preserve nonperturbative renormalisability by expanding e^{-S} in $1/M$:

$$\int D[U,\overline{\psi},\psi]\mathrm{e}^{-S} = \int D[U,\overline{\psi},\psi]\mathrm{e}^{-S_0}\left(1 + \omega_{\mathrm{kin}} O_{\mathrm{kin}} + \omega_{\mathrm{spin}} O_{\mathrm{spin}} + \dots\right)$$

where

$$S_0 = \sum_x \overline{\psi} D_0 \psi \qquad O_{\mathrm{kin}} = \sum_x \overline{\psi}\mathbf{D}^2\psi \qquad O_{\mathrm{spin}} = \sum_x \overline{\psi}\sigma\cdot\mathbf{B}\psi$$

and $\omega_{\mathrm{kin}} = \omega_{\mathrm{spin}} = 1/2M$ at tree level

- Nonperturbatively renormalisable order by order in $1/M$
- Nonperturbative matching to QCD

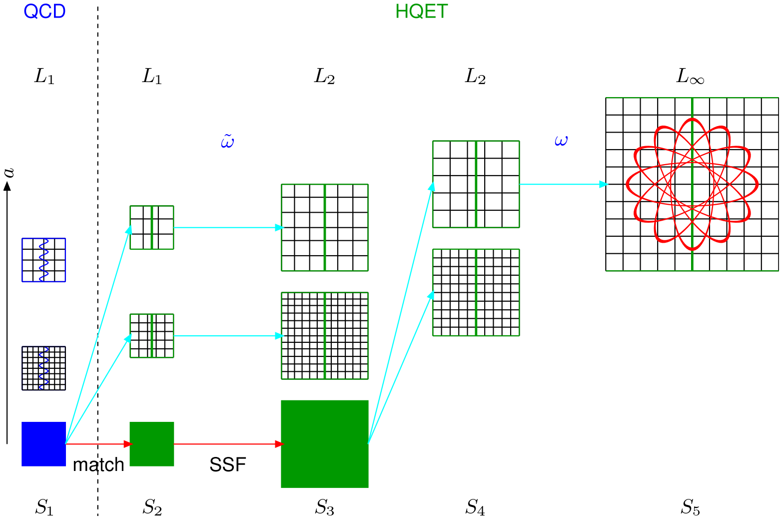
Nonperturbative HQET

- Want to determine HQET parameters by matching to QCD nonperturbatively

$$\Phi_k^{\mathrm{QCD}} = \Phi_k^{\mathrm{HQET}} \qquad k = 1, \ldots, N_{\mathrm{parms}}$$

- But that needs very small a
- Trick: make a small enough by going to small volume for matching to QCD using the Schrödinger functional
- Then scale to larger volumes in HQET by finite-size scaling (for which the continuum limit can be taken nonperturbatively) [Sommer et al. (ALPHA) 2003]
- Afterwards, can use HQET in large volume at normal lattice spacings
- Procedure has been carried out to $O(1/M)$ in quenched QCD [Della Morte et al. (ALPHA) 2007; Blossier et al. (ALPHA) 2009]
- $N_f = 2$ underway

Nonperturbative HQET



The Generalised Eigenvalue Problem

For a matrix of correlation functions on an infinite time lattice

$$C_{ij}(t) = \langle O_i(t) O_j(0) \rangle = \sum_{n=1}^{\infty} \mathrm{e}^{-E_n t} \psi_{ni} \psi_{nj} \,, \quad i,j = 1, \ldots, N$$

$$\psi_{ni} \equiv (\psi_n)_i = \langle n | \hat{O}_i | 0 \rangle = \psi_{ni}^* \quad E_n \leq E_{n+1}$$

the GEVP is

$$C(t)\, v_n(t, t_0) = \lambda_n(t, t_0)\, C(t_0)\, v_n(t, t_0) \,, \quad n = 1, \ldots, N \quad 2t_0 \geq t > t_0,$$

Define effective energy levels and matrix elements [Blossier, GvH et al.,2008]

$$E_n^{\mathrm{eff}} \equiv \frac{1}{a} \log \frac{\lambda_n(t, t_0)}{\lambda_n(t+a, t_0)} = E_n + \mathrm{O}(\mathrm{e}^{-(E_{N+1}-E_n)t})$$

$$\psi_n^{\mathrm{eff}} \equiv \frac{\langle P(t) O_j(0) \rangle\, v_n(t, t_0)_j}{(v_n(t, t_0)\,,\; C(t)\, v_n(t, t_0))^{1/2}} \frac{\lambda_n(t_0 + t/2, t_0)}{\lambda_n(t_0 + t, t_0)}$$

$$= \langle 0 | \hat{P} | n \rangle + \mathrm{O}(\mathrm{e}^{-(E_{N+1}-E_n)t_0})$$

arXiv:0911.1568

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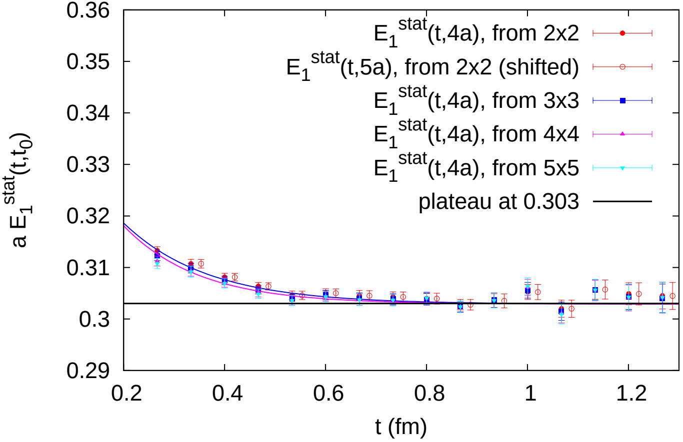

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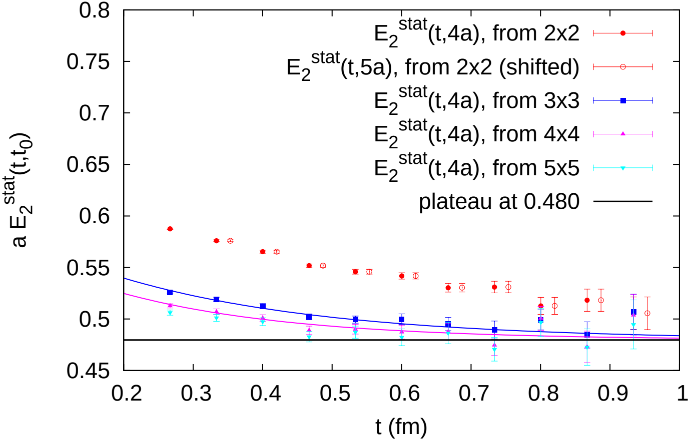


Applications in HQET

Expand in $1/M$

$$C_{ij}(t) = C_{ij}^{\mathrm{stat}}(t) + 1/M C_{ij}^{1/\mathrm{M}}(t) + \mathrm{O}(1/M^2)$$

and find to first order in $1/M$

$$E_n^{\mathrm{eff,stat}}(t,t_0) = \log \frac{\lambda_n^{\mathrm{stat}}(t,t_0)}{\lambda_n^{\mathrm{stat}}(t+1,t_0)} = E_n^{\mathrm{stat}} + \mathrm{O}(\mathrm{e}^{-\Delta E_{N+1,n}^{\mathrm{stat}}\,t}),$$

$$E_n^{\mathrm{eff},1/\mathrm{M}}(t,t_0) = \frac{\lambda_n^{1/\mathrm{M}}(t,t_0)}{\lambda_n^{\mathrm{stat}}(t,t_0)} - \frac{\lambda_n^{1/\mathrm{M}}(t+1,t_0)}{\lambda_n^{\mathrm{stat}}(t+1,t_0)}$$

$$= E_n^{1/\mathrm{M}} + \mathrm{O}(t\,\mathrm{e}^{-\Delta E_{N+1,n}^{\mathrm{stat}}\,t}).$$

where

$$C^{\mathrm{stat}}(t)\, v_n^{\mathrm{stat}}(t,t_0) = \lambda_n^{\mathrm{stat}}(t,t_0)\, C^{\mathrm{stat}}(t_0)\, v_n^{\mathrm{stat}}(t,t_0),$$

$$\lambda_n^{1/\mathrm{M}}(t,t_0) = \left(v_n^{\mathrm{stat}}(t,t_0),\, [C^{1/\mathrm{M}}(t) - \lambda_n^{\mathrm{stat}}(t,t_0) C^{1/\mathrm{M}}(t_0)] v_n^{\mathrm{stat}}(t,t_0)\right)$$

Note that the GEVP is solved only for C_{ij}^{stat}

The CLS coordinated lattice simulations effort

- CLS is a community effort to bring together the human and computer resources of several teams in Europe interested in lattice QCD
- Member teams: Berlin, CERN, DESY-Zeuthen, Madrid, Mainz, Rome, Valencia
- All CLS simulations use Lüscher's DD-HMC algorithm:
  - $N_f = 2$ Wilson QCD with non-perturbative $O(a)$ improvement
  - Domain decomposition separates block and inter-block MD forces
  - Use a Schwarz-preconditioned GCR solver on each block
  - Deflation to reduce the effort at smaller quark masses
  - Chronological solver to further accelerate MD dynamics

Results in quenched HQET at $O(1/m)$

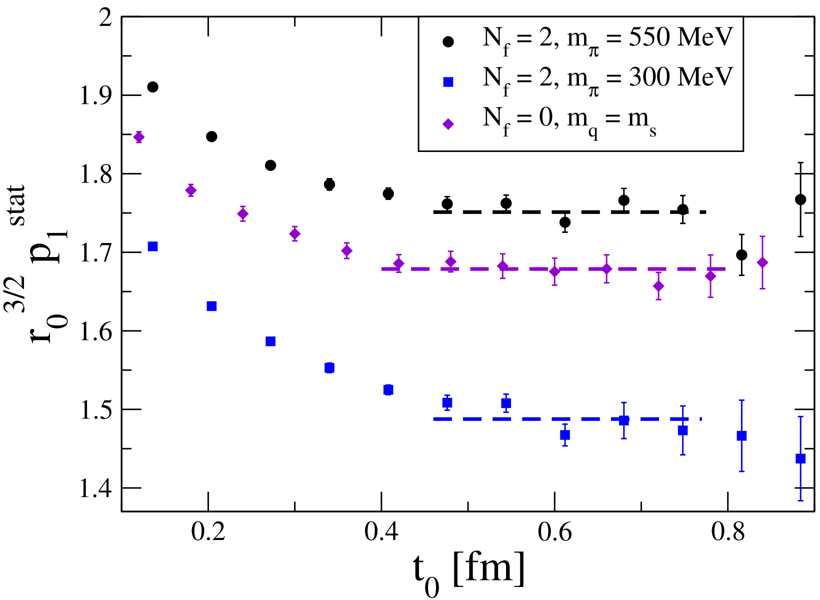
- Testing methods in quenched simulations
- Use a number of methods to reduce noise and systematic errors
  - all-to-all propagators [Foley et al. (TrinLat) 2005]
  - HYP1/HYP2 smearing for static action [Hasenfratz, Knechtli 2001; Della Morte et al. (ALPHA) 2005]
  - Covariant quark smearing with APE smeared spatial links [Güsken et al. 1989; Albanese et al. (APE) 1987; Basak et al. 2006]
  - variational method [Michael 1985; Lüscher, Wolff 1990; Blossier, GvH et al. (ALPHA) 2008]
- Take continuum limit with three lattice spacings ($\beta = 6.0219$, 6.2885, 6.4956), $L = 1.436$ fm, $T = 2L$, κ tuned to strange mass

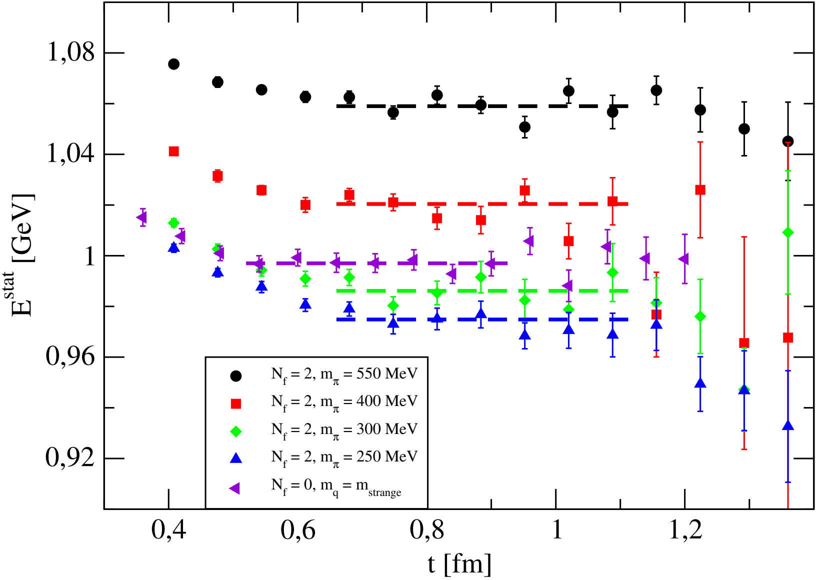
First results in $N_f = 2$ QCD

- Use noise reduction methods proven in quenched case
- Measurements on four CLS ensembles:

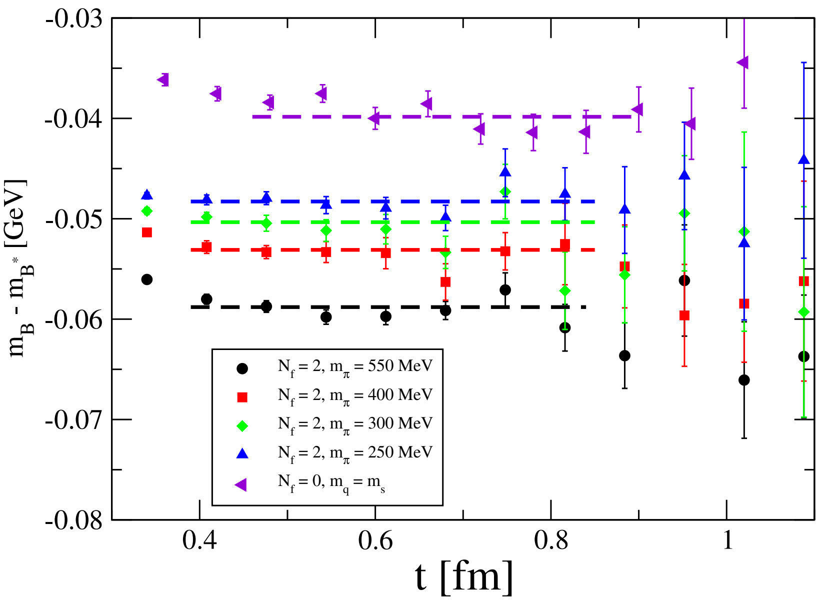
β	a [fm]	$T \times L^3$	m_π [MeV]	N_{cfg}
5.3	0.07	64×32^3	550	152
			400	600
		96×48^3	300	192
			250	350

- Take chiral limit at fixed lattice spacing using four pion masses
- Continuum limit remains to be done


N_f = 2, m_π = 550 MeV
N_f = 2, m_π = 300 MeV
N_f = 0, m_q = m_s
$r_0^{3/2}\, p_1^{\mathrm{stat}}$
t_0 [fm]
2
1.9
1.8
1.7
1.6
1.5
1.4
0.2
0.4
0.6
0.8



1,08
1,04
1
0,96
0,92
E^stat [GeV]
0,4
0,6
0,8
1
1,2
t [fm]
N_f = 2, m_π = 550 MeV
N_f = 2, m_π = 400 MeV
N_f = 2, m_π = 300 MeV
N_f = 2, m_π = 250 MeV
N_f = 0, m_q = m_strange



-0.03
-0.04
-0.05
-0.06
-0.07
-0.08
0.4
0.6
0.8
1
t [fm]
$m_B - m_{B^*}$ [GeV]
$N_f = 2$, $m_\pi = 550$ MeV
$N_f = 2$, $m_\pi = 400$ MeV
$N_f = 2$, $m_\pi = 300$ MeV
$N_f = 2$, $m_\pi = 250$ MeV
$N_f = 0$, $m_q = m_s$

arXiv:1004.2661

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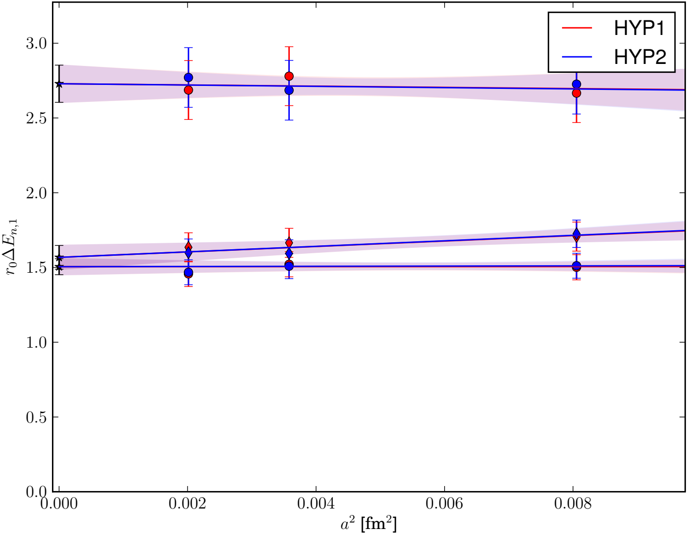

arXiv:1004.2661

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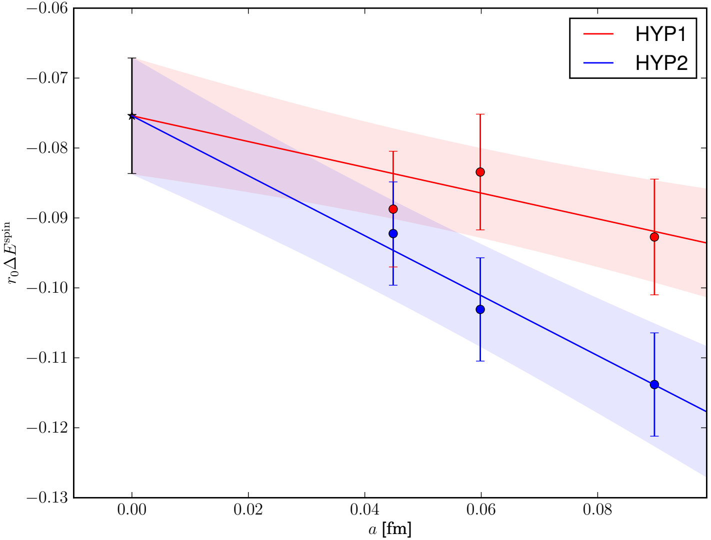

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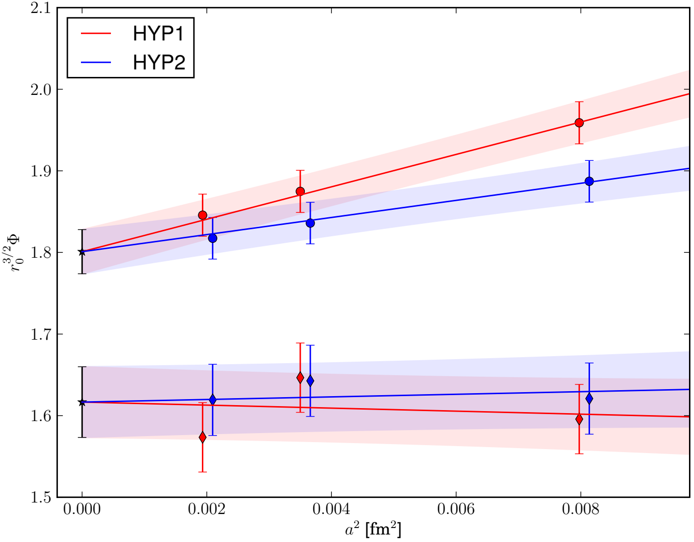
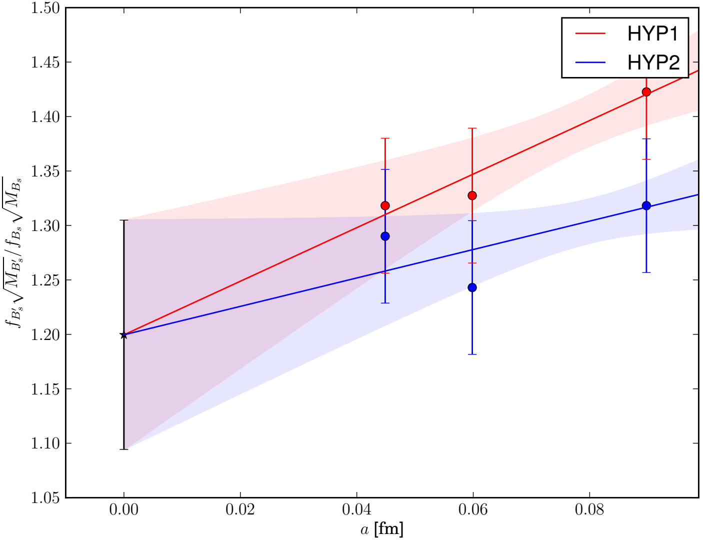


PRELIMINARY



How far can HQET go?

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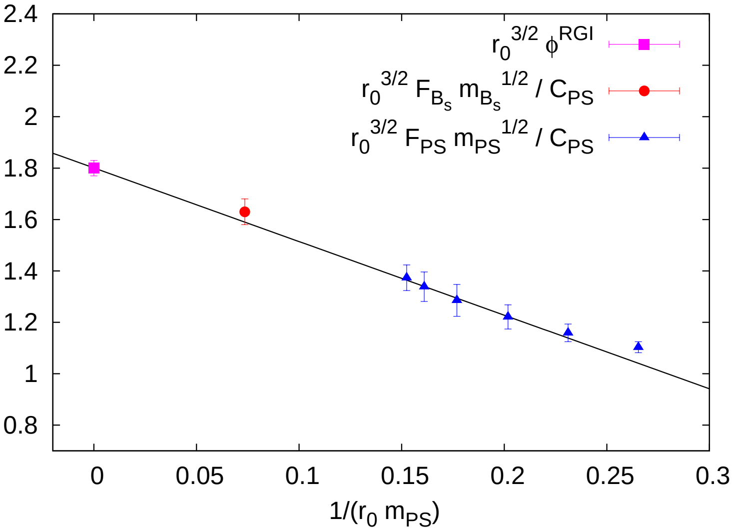
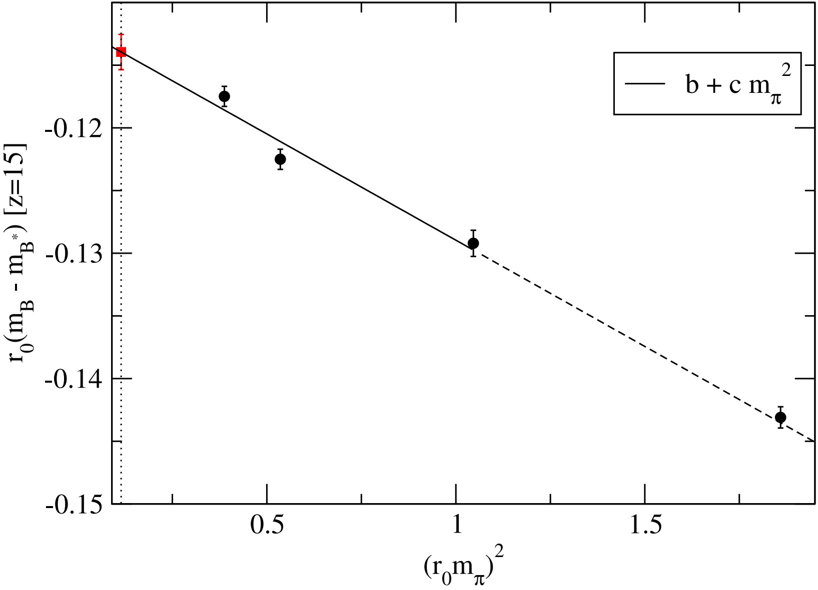
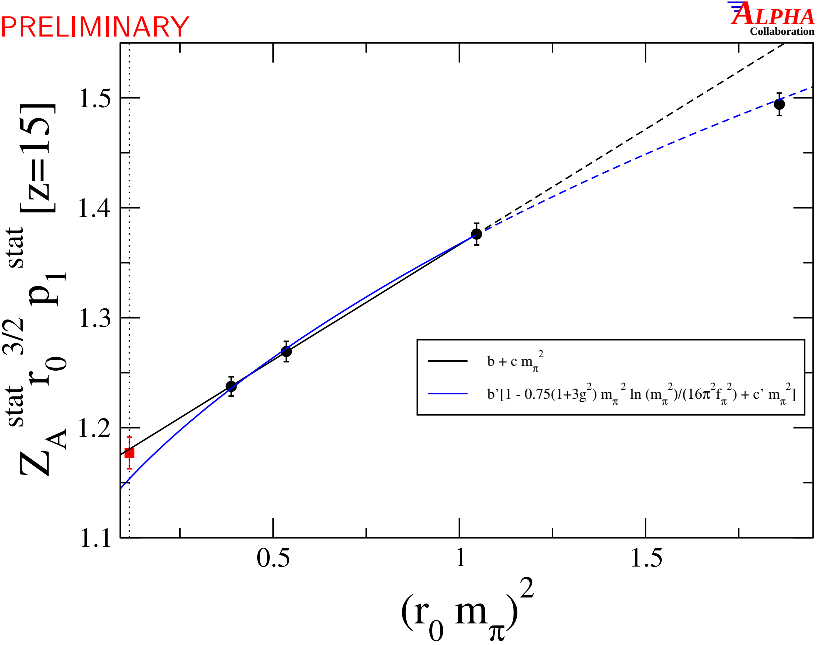


PRELIMINARY

ALPHA Collaboration



PRELIMINARY



PRELIMINARY

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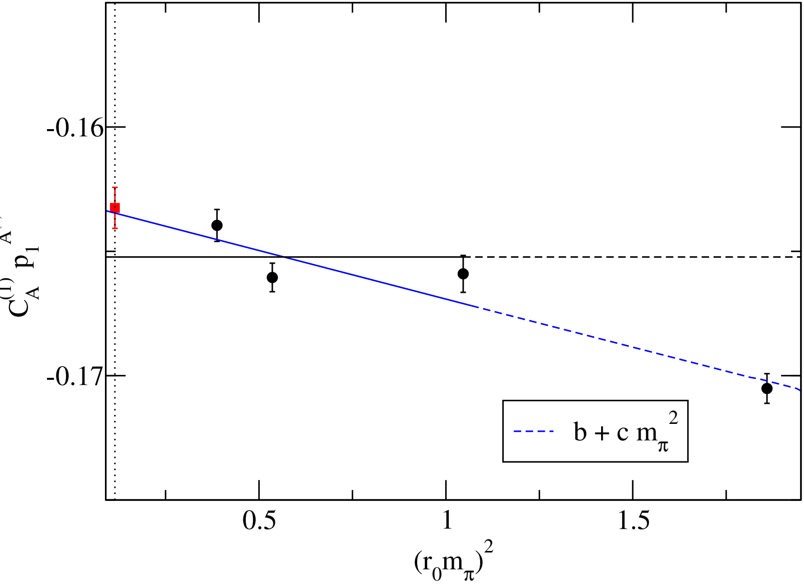


Outlook for heavy flavours on the lattice

- Large fine dynamical lattices still difficult
- Effective field theories provide a rigorous, systematically improvable way of studying heavy quarks on the lattice
- HQET works well for bottom quarks
- HQET may perhaps even work for charm quarks
- $N_f = 0$ HQET matched nonperturbatively
- $N_f = 2$ nonperturbative HQET well under way

The end

Thank you for your attention

Backup slides

— BACKUP —

The CKM formalism

Cabibbo (1963): Mixing of quarks to explain decay patterns:

$$\begin{bmatrix} |d'\rangle \\ |s'\rangle \end{bmatrix} = \begin{bmatrix} V_{ud} & V_{us} \\ V_{cd} & V_{cs} \end{bmatrix} \begin{bmatrix} |d\rangle \\ |s\rangle \end{bmatrix} = \begin{bmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{bmatrix} \begin{bmatrix} |d\rangle \\ |s\rangle \end{bmatrix}$$

Kobayashi, Maskawa (1973, Nobel 2008): Extension to three generations predicts CP-violation

$$\begin{aligned} \begin{bmatrix} |d'\rangle \\ |s'\rangle \\ |b'\rangle \end{bmatrix} &= \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \begin{bmatrix} |d\rangle \\ |s\rangle \\ |b\rangle \end{bmatrix} \\ &= \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix} \begin{bmatrix} |d\rangle \\ |s\rangle \\ |b\rangle \end{bmatrix} \end{aligned}$$

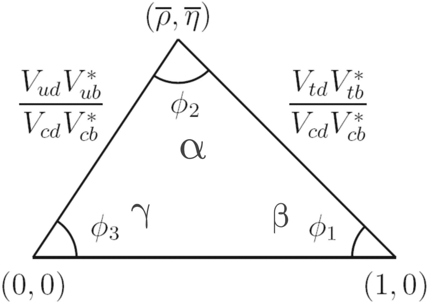
The Unitarity Triangle

Unitarity requires columns to be orthogonal, e.g.

$$V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0$$

or equivalently

$$\frac{V_{ud} V_{ub}^*}{V_{cd} V_{cb}^*} + \frac{V_{td} V_{tb}^*}{V_{cd} V_{cb}^*} = -1$$



The Generalised Eigenvalue Problem

For a matrix of correlation functions on an infinite time lattice

$$C_{ij}(t) = \langle O_i(t) O_j(0) \rangle = \sum_{n=1}^{\infty} \mathrm{e}^{-E_n t} \psi_{ni} \psi_{nj} \,, \quad i,j = 1, \ldots, N$$

$$\psi_{ni} \equiv (\psi_n)_i = \langle n | \hat{O}_i | 0 \rangle = \psi_{ni}^* \quad E_n \leq E_{n+1}$$

the GEVP is

$$C(t)\, v_n(t,t_0) = \lambda_n(t,t_0)\, C(t_0)\, v_n(t,t_0) \,, \quad n = 1, \ldots, N \quad t > t_0 ,$$

Define effective energy levels and matrix elements[Blossier, GvH et al.,2008]

$$E_n^{\mathrm{eff}} \equiv \frac{1}{a} \log \frac{\lambda_n(t,t_0)}{\lambda_n(t+a,t_0)} = E_n + \mathrm{O}(\mathrm{e}^{-(E_{N+1}-E_n)t})$$

$$\psi_n^{\mathrm{eff}} \equiv \frac{\langle P(t) O_j(0) \rangle \, v_n(t,t_0)_j}{(v_n(t,t_0)\,,\; C(t)\, v_n(t,t_0))^{1/2}} \frac{\lambda_n(t_0+t/2,t_0)}{\lambda_n(t_0+t,t_0)}$$

$$= \langle 0 | \hat{P} | n \rangle + \mathrm{O}(\mathrm{e}^{-(E_{N+1}-E_n)t_0})$$

The Generalised Eigenvalue Problem

Can construct creation operator

$$
\begin{aligned}
\hat{\mathcal{A}}_n^{\mathrm{eff}\,\dagger}(t,t_0) &= \mathrm{e}^{-\hat{H}t}\,\hat{\mathcal{Q}}_n^{\mathrm{eff}\,\dagger}(t,t_0) \\
\hat{\mathcal{Q}}_n^{\mathrm{eff}}(t,t_0) &= R_n\,(\hat{O}\,,\ v_n(t,t_0)\,) \\
R_n &= (v_n(t,t_0)\,,\ C(t)\,v_n(t,t_0))^{-1/2}\,\frac{\lambda_n(t_0+t/2,t_0)}{\lambda_n(t_0+t,t_0)}
\end{aligned}
$$

for n^{th} excited state:

$$
\hat{\mathcal{A}}_n^{\mathrm{eff}\,\dagger}|0\rangle = |n\rangle + \sum_{n'=1}^{\infty} \pi_{nn'}(t,t_0)|n'\rangle
$$

where

$$
\pi_{nn'}(t,t_0) = \mathrm{O}(\mathrm{e}^{-(E_{N+1}-E_n)\,t_0})\,, \quad \text{at fixed } t-t_0
$$

The GEVP simplified

(Theoretically) split C_{ij} into first N states and the rest

$$C^{(0)}_{ij}(t) = \sum_{n=1}^{N} \mathrm{e}^{-E_n t}\psi_{ni}\psi_{nj}\,, \qquad C^{(1)}_{ij}(t) = \sum_{n=N+1}^{\infty} \mathrm{e}^{-E_n t}\psi_{ni}\psi_{nj}$$

The (time-independent) dual vectors are defined by

$$(u_n, \psi_m) = \delta_{mn}\,, \;\; m, n \leq N\,. \qquad (u_n, \psi_m) \equiv \sum_{i=1}^{N} (u_n)_i\,\psi_{mi}$$

One then has

$$\begin{aligned} C^{(0)}(t) u_n &= \mathrm{e}^{-E_n t}\psi_n\,, \\ C^{(0)}(t)\, u_n &= \lambda_n^{(0)}(t, t_0)\, C^{(0)}(t_0)\, u_n, \\ \lambda_n^{(0)}(t, t_0) &= \mathrm{e}^{-E_n(t-t_0)}\,, \quad v_n(t, t_0) \propto u_n \end{aligned}$$

and an orthogonality relation valid at all t

$$(u_m, C^{(0)}(t)\, u_n) = \delta_{mn}\,\rho_n(t)\,, \quad \rho_n(t) = \mathrm{e}^{-E_n t}\,.$$

The GEVP simplified

The operators

$$\hat{\mathcal{A}}_n = \sum_{i=1}^{N} (u_n)_i \hat{O}_i \equiv (\hat{O}, u_n)\,,$$

create the eigenstates of the Hamilton operator

$$|n\rangle = \hat{\mathcal{A}}_n |0\rangle\,, \hat{H}|n\rangle = E_n\,|n\rangle\,.$$

So arbitrary matrix elements can be written as

$$p_{0n} \quad = \quad \langle 0|\hat{P}|n\rangle = \langle 0|\hat{P}\hat{\mathcal{A}}_n|0\rangle$$

generalization:

$$\begin{aligned} p_{0n} &= \langle P(t) O_j(0)\rangle (u_n)_j = \frac{\langle P(t)\mathcal{A}_n(0)\rangle}{\langle \mathcal{A}_n(t)\mathcal{A}_n(0)\rangle^{1/2}} \mathrm{e}^{E_n\,t/2} \\ &= \frac{\langle P(t) O_j(0)\rangle\, v_n(t,t_0)_j}{(v_n(t,t_0)\,,\ C(t)\, v_n(t,t_0))^{1/2}} \frac{\lambda_n(t_0+t/2,t_0)}{\lambda_n(t_0+t,t_0)} \end{aligned}$$

Perturbation theory for the GEVP

Following [Ferenc Niedermayer & Peter Weisz, 1998, unpublished] we set up the perturbative expansion for the GEVP as

$$A v_n = \lambda_n B v_n \,, \quad A = A^{(0)} + \epsilon A^{(1)} \,, \quad B = B^{(0)} + \epsilon B^{(1)} \,.$$

$$(v_n^{(0)}, B^{(0)} v_m^{(0)}) \quad = \quad \rho_n \, \delta_{nm} \,.$$

$$\begin{aligned} \lambda_n &= \lambda_n^{(0)} + \epsilon \lambda_n^{(1)} + \epsilon^2 \lambda_n^{(2)} \, \ldots \\ v_n &= v_n^{(0)} + \epsilon v_n^{(1)} + \epsilon^2 v_n^{(2)} \, \ldots \end{aligned}$$

We will later set

$$\begin{aligned} A^{(0)} &= C^{(0)}(t) \,, \quad \epsilon A^{(1)} = C^{(1)}(t) \,, \\ B^{(0)} &= C^{(0)}(t_0) \,, \quad \epsilon B^{(1)} = C^{(1)}(t_0) \end{aligned}$$

Perturbation theory for the GEVP

To second order

$$
\begin{aligned}
A^{(0)} v_n^{(1)} + A^{(1)} v_n^{(0)} &= \lambda_n^{(0)} \left[B^{(0)} v_n^{(1)} + B^{(1)} v_n^{(0)} \right] + \lambda_n^{(1)} B^{(0)} v_n^{(0)} \, , \\
A^{(0)} v_n^{(2)} + A^{(1)} v_n^{(1)} &= \lambda_n^{(0)} \left[B^{(0)} v_n^{(2)} + B^{(1)} v_n^{(1)} \right] + \lambda_n^{(1)} \left[B^{(0)} v_n^{(1)} + B^{(1)} v_n^{(0)} \right] + \lambda_n^{(2)} B^{(0)} v_n^{(0)} \, .
\end{aligned}
$$

Solve using orthogonality $(v_n^{(0)}, B^{(0)} v_m^{(0)}) = \delta_{mn} \rho_n$

$$
\begin{aligned}
\lambda_n^{(1)} &= \rho_n^{-1} \left(v_n^{(0)}, \Delta_n v_n^{(0)} \right) , \quad \Delta_n \equiv A^{(1)} - \lambda_n^{(0)} B^{(1)} \\
v_n^{(1)} &= \sum_{m \neq n} \alpha_{nm}^{(1)} \rho_m^{-1/2} v_m^{(0)} , \quad \alpha_{nm}^{(1)} = \rho_m^{-1/2} \frac{\left(v_m^{(0)}, \Delta_n v_n^{(0)} \right)}{\lambda_n^{(0)} - \lambda_m^{(0)}} \\
\lambda_n^{(2)} &= \sum_{m \neq n} \rho_n^{-1} \rho_m^{-1} \frac{\left(v_m^{(0)}, \Delta_n v_n^{(0)} \right)^2}{\lambda_n^{(0)} - \lambda_m^{(0)}} - \rho_n^{-2} \left(v_n^{(0)}, \Delta_n v_n^{(0)} \right) \left(v_n^{(0)}, B^{(1)} v_n^{(0)} \right) \, .
\end{aligned}
$$

Also find an all-orders recursion formula for the higher order coefficients

Perturbation theory for the GEVP

Inserting our specific case and using (for $m > n$)

$$\begin{aligned}(\lambda_n^{(0)} - \lambda_m^{(0)})^{-1} &= (\lambda_n^{(0)})^{-1}(1 - \mathrm{e}^{-(E_m - E_n)(t-t_0)})^{-1} \\ &= (\lambda_n^{(0)})^{-1} \sum_{k=0}^{\infty} \mathrm{e}^{-k(E_m-E_n)(t-t_0)}\end{aligned}$$

we get

$$\begin{aligned}\varepsilon_n(t,t_0) &= \mathrm{O}(\mathrm{e}^{-\Delta E_{N+1,n}\, t})\,, \quad \Delta E_{m,n} = E_m - E_n\,, \\ \pi_{nn'}(t,t_0) &= \mathrm{O}(\mathrm{e}^{-\Delta E_{N+1,n}\, t_0})\,, \quad \text{at fixed } t - t_0 \\ \pi_1(t,t_0) &= \mathrm{O}(\mathrm{e}^{-\Delta E_{N+1,1}\, t_0}\mathrm{e}^{-\Delta E_{2,1}\,(t-t_0)}) + \mathrm{O}(\mathrm{e}^{-\Delta E_{N+1,1}\, t})\,.\end{aligned}$$

to all orders in the perturbative expansion

Applications in HQET

Expand in $1/M$

$$C_{ij}(t) \quad = \quad C_{ij}^{\rm stat}(t) + 1/M C_{ij}^{1/{\rm M}}(t) + {\rm O}(1/M^2)$$

and find to first order in $1/M$

$$E_n^{\rm eff,stat}(t,t_0) \quad = \quad \log \frac{\lambda_n^{\rm stat}(t,t_0)}{\lambda_n^{\rm stat}(t+1,t_0)} = E_n^{\rm stat} + {\rm O}({\rm e}^{-\Delta E_{N+1,n}^{\rm stat}\, t})\,,$$

$$E_n^{\rm eff,1/M}(t,t_0) \quad = \quad \frac{\lambda_n^{1/{\rm M}}(t,t_0)}{\lambda_n^{\rm stat}(t,t_0)} - \frac{\lambda_n^{1/{\rm M}}(t+1,t_0)}{\lambda_n^{\rm stat}(t+1,t_0)}$$

$$\quad = \quad E_n^{1/{\rm M}} + {\rm O}(t\,{\rm e}^{-\Delta E_{N+1,n}^{\rm stat}\, t})\,.$$

where

$$C^{\rm stat}(t)\, v_n^{\rm stat}(t,t_0) = \lambda_n^{\rm stat}(t,t_0)\, C^{\rm stat}(t_0)\, v_n^{\rm stat}(t,t_0)\,,$$

$$\lambda_n^{1/{\rm M}}(t,t_0) \quad = \quad \Big(v_n^{\rm stat}(t,t_0)\,,\,[C^{1/{\rm M}}(t) - \lambda_n^{\rm stat}(t,t_0) C^{1/{\rm M}}(t_0)]\, v_n^{\rm stat}(t,t_0)\Big)$$

Note that the GEVP is solved only for $C_{ij}^{\rm stat}$