

SYK模型と多体局在

KEK「素核宇・物性」連携研究会

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Plan

- Sachdev-Ye-Kitaev model

$$\hat{H} = \sum_{1 \leq a < b < c < d \leq N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

- Maximally chaotic quantum mechanical model

- SYK4+2

$$\hat{H} = \sum_{1 \leq a < b < c < d}^N J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d + i \sum_{1 \leq a < b}^N K_{ab} \hat{\chi}_a \hat{\chi}_b$$

- Departure from chaotic behavior

- Quantitative analysis of Fock-space localization

- Many-body transition point
- Inverse participation ratio
- Entanglement entropy

Publications and collaborators

- Sachdev-Ye-Kitaev model
 - Proposal for experiment: PTEP 2017, 083I01 and arXiv:1709.07189
 - with Ippei Danshita and Masanori Hanada
 - Black Holes and Random Matrices: JHEP 1705(2017)118
 - with J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, and A. Streicher
- SYK4+2
 - Chaotic-integrable transition: PRL **120**, 241603 (2018)
 - with Antonio M. García-García, Bruno Loureiro, and Aurelio Romero-Bermúdez
 - Characterization of quantum chaos: JHEP 1904(2019)082 and Phys. Rev. E **102**, 022213 (2020)
 - with Hrant Gharibyan, M. Hanada, and Brian Swingle
 - Related setups:
 - [short-range interactions] Phys. Rev. B **99**, 054202 (2019) with A. M. García-García
 - Phys. Lett. B **795**, 230 (2019) and J. Phys. A **54**, 095401 (2021) with Pak Hang Chris Lau, Chen-Te Ma, and Jeff Murugan
- Quantitative analysis of Fock-space localization in SYK4+2
 - Many-body transition point and inverse participation ratio
 - Phys. Rev. Research **3**, 013023 (2021) with Felipe Monteiro, Tobias Micklitz, and Alexander Altland
 - Entanglement entropy
 - arXiv:2012.07884 with F. Monteiro, A. Altland, David A. Huse, and T. Micklitz

Sachdev-Ye-Kitaev (SYK) 模型

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \leq a < b < c < d \leq N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

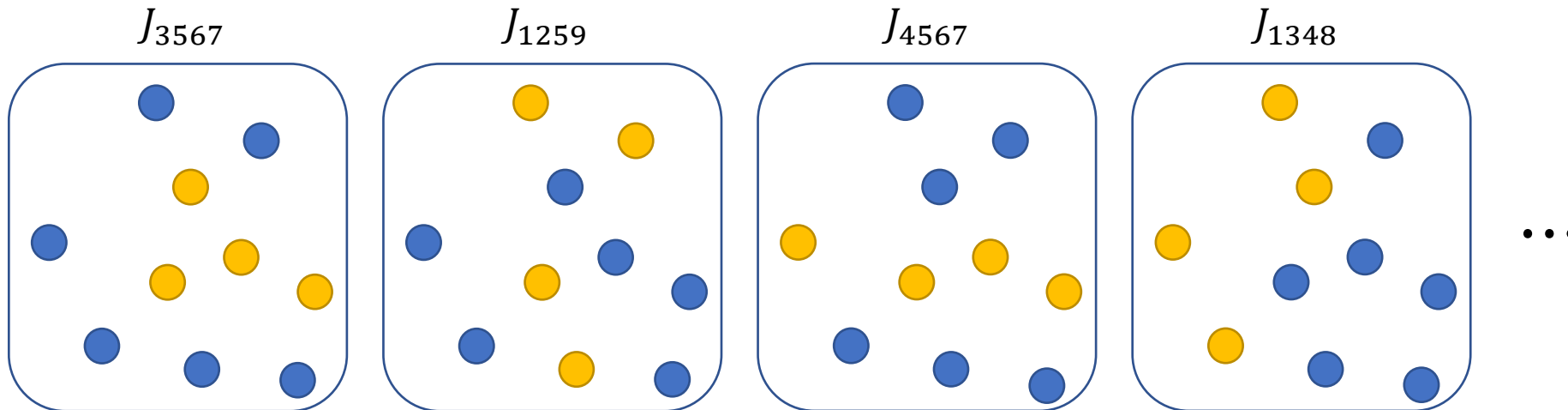
[A. Kitaev: talks at KITP

(Feb 12, Apr 7 and May 27, 2015)]

cf. SY model [S. Sachdev and J. Ye, 1993]

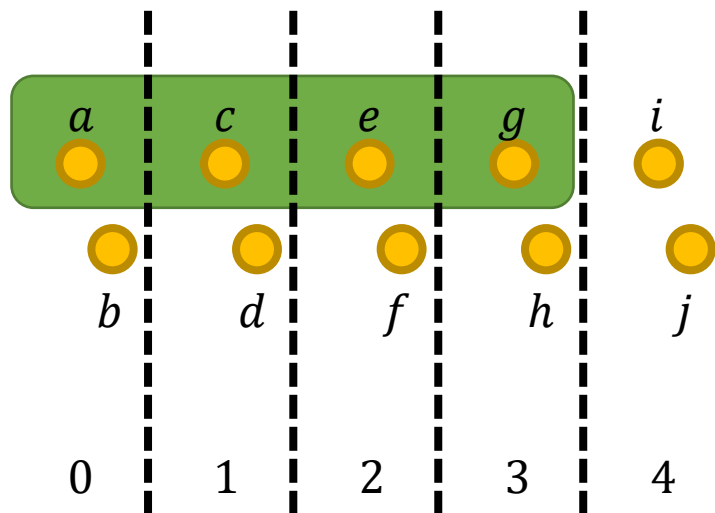
$\hat{\chi}_{a=1,2,\dots,N}$: N 個のマヨラナフェルミオン ($\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$)

J_{abcd} : 独立にガウス分布する結合定数 ($\overline{J_{abcd}^2} = J^2 (= 1)$, $\overline{J_{abcd}} = 0$)



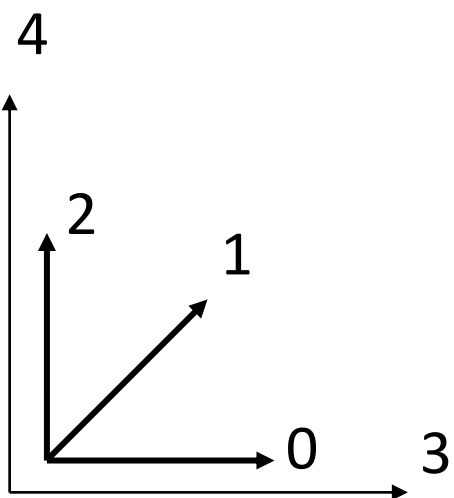
One term of the 10-Majorana fermion SYK_{q=4}

$\chi_a \chi_c \chi_e \chi_g$

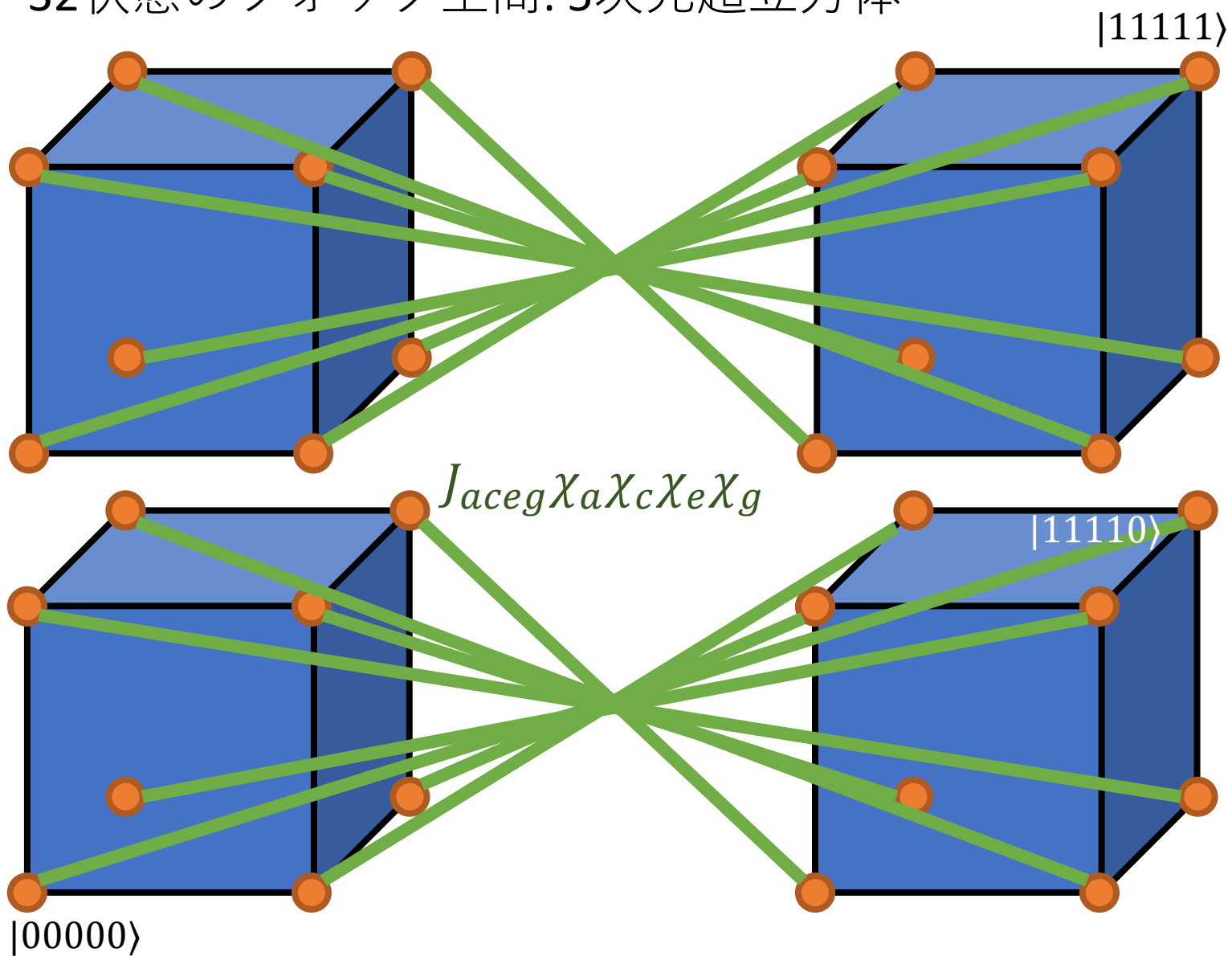


5 qubits

$\binom{10}{4} = 210$ 個の項



32状態のフォック空間: 5次元超立方体



Sachdev-Ye-Kitaev 模型

N 個の（マヨラナまたはディラック）フェルミオン、ランダム全対全結合

[Majorana version]

$$\hat{H} = \frac{\sqrt{3}!}{N^{3/2}} \sum_{1 \leq a < b < c < d \leq N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

[A. Kitaev: talks at KITP (2015)]

[Dirac version]

$$\hat{H} = \frac{1}{(2N)^{3/2}} \sum_{ij;kl} J_{ij;kl} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l$$

[A. Kitaev's talk]

[S. Sachdev: PRX **5**, 041025 (2015)]

cf. SY model [S. Sachdev and J. Ye, 1993]

原子核理論に関係して、古くから研究されていた

- [French and Wong, Phys. Lett. B **33**, 449 (1970)]
- [Bohigas and Flores, Phys. Lett. B **34**, 261 (1971)]

“Two-body Random Ensemble”

N : フェルミオンの個数

SYK: $N \gg 1$ で解けるモデル (サンプル平均 $\langle \dots \rangle_{\{J\}}$ が前提)

非摂動ハミルトニアン = 0,

$$\hat{H} = \frac{\sqrt{3}!}{N^{3/2}} \sum_{1 \leq a < b < c < d \leq N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

を摂動とみてダイアグラム展開

$\langle J_{abcd}^2 \rangle_{\{J\}} = J^2$, ガウス分布

$\langle J_{abcd} J_{abce} \rangle_{\{J\}} = 0$ if $d \neq e \rightarrow$ 大部分のダイアグラムの平均は 0

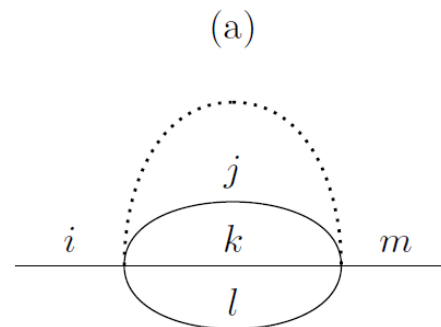
自由粒子の2点関数

$$\begin{aligned} G_{0,ij}(t) &= -\langle T \psi_i(t) \psi_j(0) \rangle \\ &= -\frac{1}{2} \text{sgn}(t) \delta_{ij} \end{aligned}$$

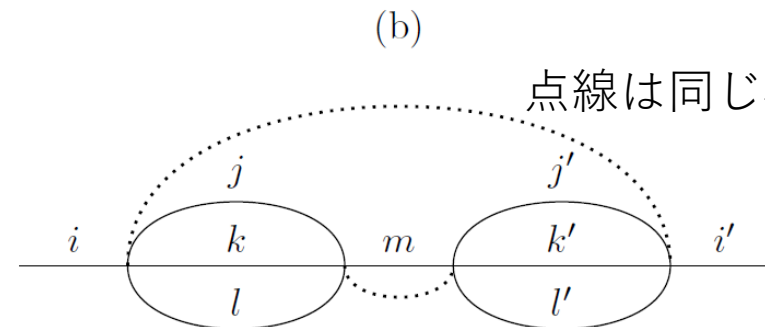
サンプル平均をとると、「メロン型」ダイアグラムのみ生き残る



From Wikimedia Commons (By Aravind Sivaraj (2012))
CC BY-SA 3.0



$O(1)$ melon



$O(N^{-2})$ not melon 🍈

点線は同じ相互作用をつなぐ

ダイアグラム展開

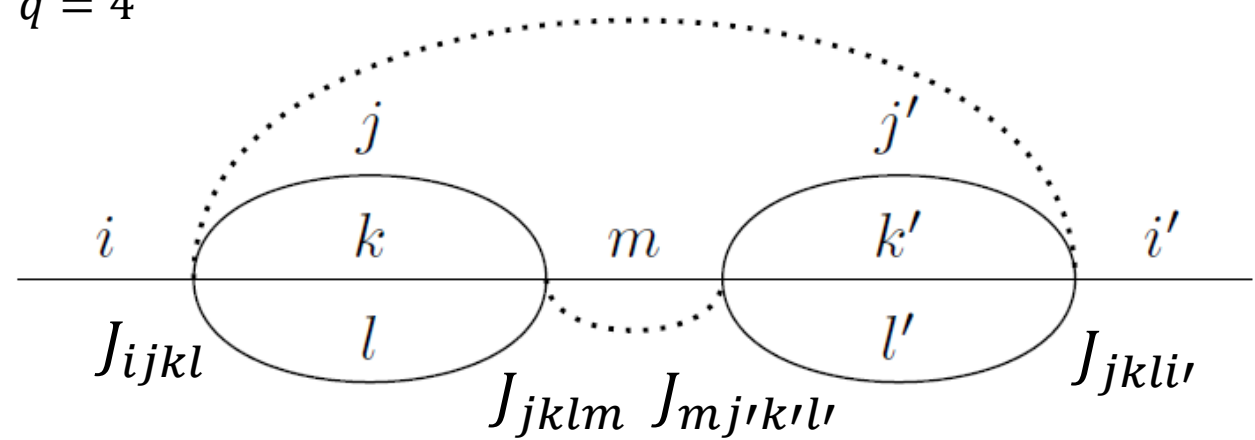
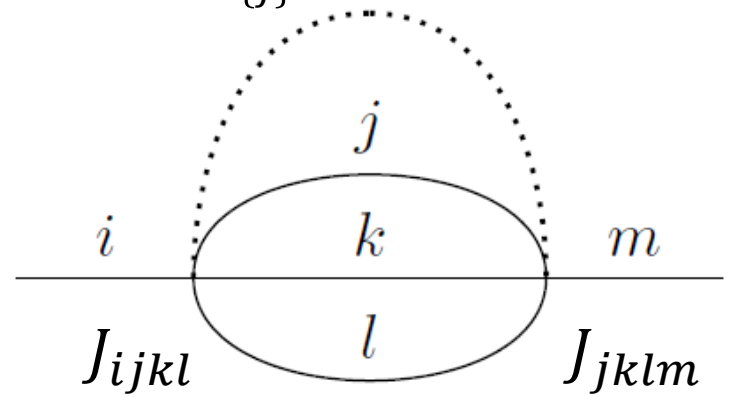
[J. Polchinski and V. Rosenhaus, JHEP 1604 (2016) 001]
 [J. Maldacena and D. Stanford, PRD **94**, 106002 (2016)]

$$\hat{H}_{\text{SYK4}} = \frac{\sqrt{3}!}{N^{3/2}} \sum_{1 \leq a < b < c < d \leq N} J_{abcd} \underbrace{\hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d}_{q=4}$$

$$\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$$

$$\langle J_{abcd} \rangle^2 = J^2 = 1$$

サンプル平均 $\langle \dots \rangle_{\{J\}}$



$$\sum_{jkl} \langle J_{ijkl} J_{jklm} \rangle_{\{J\}} = \frac{N^3}{3!} \delta_{im}$$

$$\sum_{m \neq i} \sum_{jklj'k'l'} \langle J_{ijkl} J_{jklm} J_{mj'k'l'} J_{j'k'l'i'} \rangle_{\{J\}} \propto N^4 \delta_{ii'}$$

$\Rightarrow O(N^0)$ の寄与

$\Rightarrow O(N^{-2})$ の寄与

Large- N : 「メロン型」ダイアグラムが支配的

$N \gg 1$ 極限で支配的なダイアグラム

$$\begin{aligned}
 G = & \text{---} + \underbrace{\text{---} \circ \text{---} + \text{---} \circ \text{---} \circ \text{---} + \text{---} \circ \text{---} \circ \text{---} \circ \text{---}}_{G^0 \Sigma G^0} + \dots \\
 & + \underbrace{\text{---} \circ \text{---} \circ \text{---} + \text{---} \circ \text{---} \circ \text{---} \circ \text{---}}_{G^0 \Sigma G^0 \Sigma G^0} + \dots \\
 & + \underbrace{\text{---} \circ \text{---} \circ \text{---} \circ \text{---} + \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---}}_{G^0 \Sigma G^0 \Sigma G^0 \Sigma G^0 + \dots} = G^0 (1 - \Sigma G^0)^{-1}
 \end{aligned}$$

$\nwarrow$ G^0

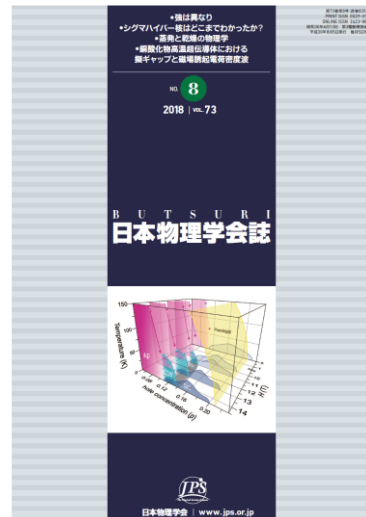
[Sachdev and Ye 1993],
[Parcollet and Georges 1999], ...

$$G(1 - \Sigma G_0) = G_0$$

$$G^{-1} = G_0^{-1} - \Sigma$$

$$G(i\omega)^{-1} = i\omega - \Sigma(i\omega)$$

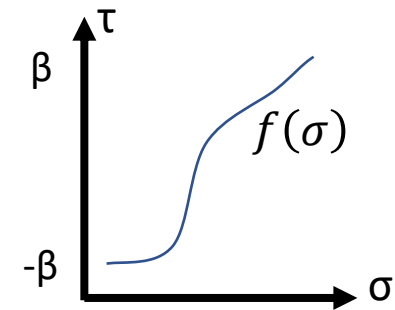
↑ 図は[I. Danshita, M. Tezuka, and M. Hanada: Butsuri **73**(8), 569 (2018)] より →



パラメータの取り換えの自由度

$$G(i\omega)^{-1} = i\omega - \Sigma(i\omega) \quad \text{低エネルギー } (\omega, T \ll J): i\omega \text{ を無視}$$

$$\int d\tau_2 G(\tau_1, \tau_2) \tilde{\Sigma}(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3),$$
$$\tilde{\Sigma}(\tau_1, \tau_2) = -J^2 [G(\tau_1, \tau_2)]^2 G(\tau_2, \tau_1)$$



虚時間のパラメータの取り換えで不変

$$\tau = f(\sigma),$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1, \sigma_2),$$

$$\tilde{\Sigma}(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \tilde{\Sigma}(\sigma_1, \sigma_2),$$

f, g は単調で微分可能な任意関数

emergent conformal gauge invariance

[S. Sachdev, Phys. Rev. X **5**, 041025 (2015)]

“System **nearly** invariant under a full reparametrization (Virasoro) symmetry, NCFT_1 ”

[J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016)]
Study of the Goldstone modes: e.g. [D. Bagrets, A. Altland, and A. Kamenev, Nucl. Phys. B **911**, 191 (2016)]

鞍点解での対称性の破れ

Large- N での鞍点解 (レプリカ対称性を仮定)

S. Sachdev, Phys. Rev. X **5**, 041025 (2015);
J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016)
Antal Jevicki, Kenta Suzuki, and Junggi Yoon,
JHEP07(2016)007

$$G_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-1/2} \quad , \quad \Sigma_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-3/2}$$

パラメータの取り換えの自由度は以下のものに限定される:

$$f(\tau) = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1.$$

1+1次元重力と双対な模型に期待されるように、
対称性は $SL(2, R)$ に破れる cf. isometry group of AdS_2
[see e.g. A. Strominger, hep-th/9809027]

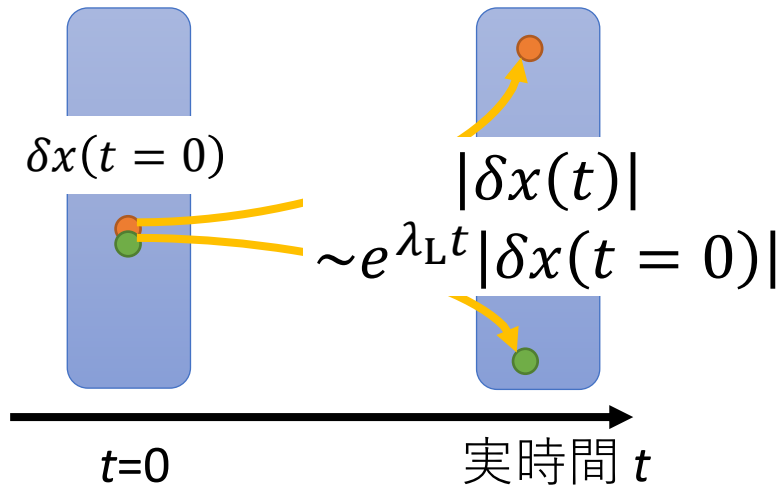
Jackiw-Teitelboim (JT) 重力: 極限に近いブラックホールの
地平線近傍での1+1次元ディラトン重力

カオスを特徴づけるリアプノフ指数の、 非時間順序相関 OTOC からの定義

$$F(t) = \langle \hat{W}^\dagger(t) \hat{V}^\dagger(0) \hat{W}(t) \hat{V}(0) \rangle \quad W(t) = e^{iHt} W e^{-iHt}$$

古典カオス:

微小にずれた初期値から時間発展



λ_L : リアプノフ指数

$$\left(\frac{\partial x(t)}{\partial x(0)} \right)^2 = \{x(t), p(0)\}_{\text{PB}}^2 \rightarrow e^{2\lambda_L t}$$

量子系の時間発展:

$$C_T(t) = \langle [\hat{x}(t), \hat{p}(0)]^2 \rangle$$

演算子 V と W について、

$$C(t) = \langle |[W(t), V(t=0)]|^2 \rangle = \langle W^\dagger(t) V^\dagger(0) W(t) V(0) \rangle + \dots$$

[Wiener 1938][Larkin & Ovchinnikov 1969]

長時間で $\text{OTOC} \sim e^{2\lambda_L t}$, $\lambda_L > 0$: カオス

「ブラックホールは最速のスクランブラーである」

[P. Hayden and J. Preskill 2007] [Y. Sekino and L. Susskind 2008]

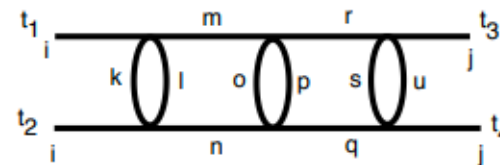
[Shenker and Stanford 2014]

$\lambda_L \leq 2\pi k_B T / \hbar$ (カオスの上限)

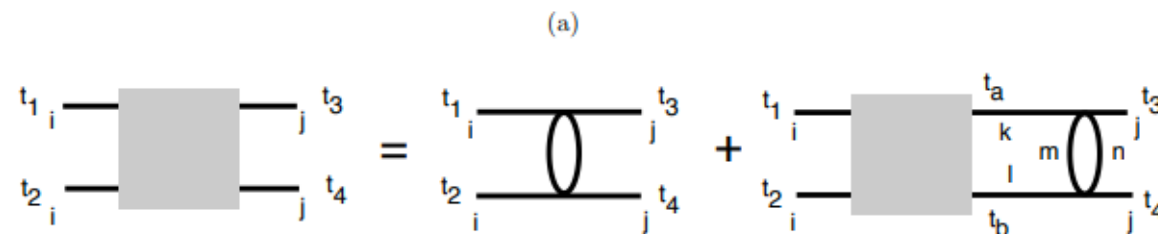
[J. Maldacena, S. H. Shenker, and D. Stanford, JHEP08(2016)106]

非時間順序相関 (OTOC)

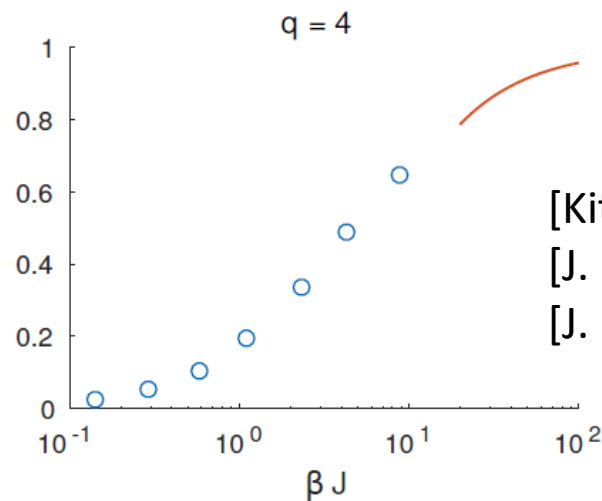
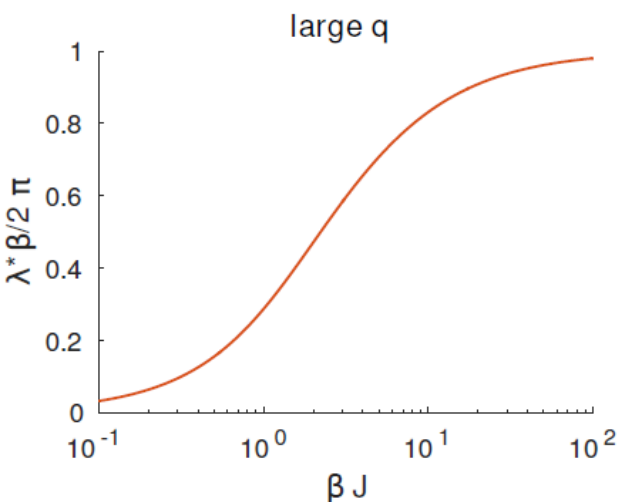
$$\langle \hat{\chi}_i(t_1) \hat{\chi}_i(t_2) \hat{\chi}_j(t_3) \hat{\chi}_j(t_4) \rangle$$



(正則化された) OTOC が large- N SYK 模型で計算でき、低温でカオスの上限 $\lambda_L = 2\pi k_B T / \hbar$ を満たす



$$\Gamma(t_1, t_2, t_3, t_4) = \Gamma_0(t_1, t_2, t_3, t_4) + \int dt_a dt_b \Gamma(t_1, t_2, t_a, t_b) K(t_a, t_b, t_3, t_4)$$



[Kitaev's talks]

[J. Polchinski and V. Rosenhaus, JHEP 1604 (2016) 001]

[J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016)]

カオスの極限にある系

S. Sachdev, Phys. Rev. Lett. 105, 151602 (2010),
Phys. Rev. X 5, 041025 (2015);
J. Maldacena and D. Stanford,
Phys. Rev. D 94, 106002 (2016); ...

0+1次元 SY &
SYK 模型

J. S. Cotler, G. Gur-Ari, M. Hanada, J.
Polchinski, P. Saad, S. H. Shenker, D.
Stanford, A. Streicher, and MT, JHEP
1705(2017)118; Y. Jia and J. J. M.
Verbaarschot, JHEP 2007(2020)193; ...

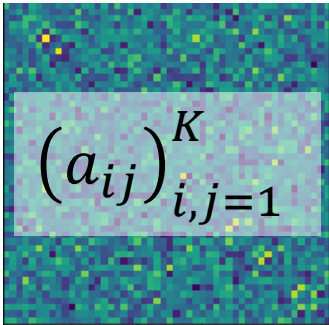
1+1次元
JT 重力

ランダム
行列

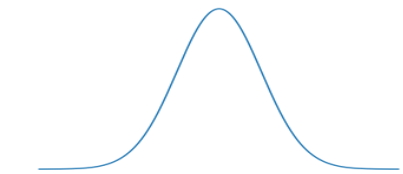
P. Saad, S. H. Shenker, and D. Stanford, arXiv:1903.11115;
D. Stanford and E. Witten, arXiv:1907.03363; ...

ガウシアンランダム行列理論

[Fidkowski and Kitaev 2010]
[You, Ludwig, and Xu 2017]
対応するSYK模型
(マヨラナ4体版)



$a_{ij} = a_{ji}^*$

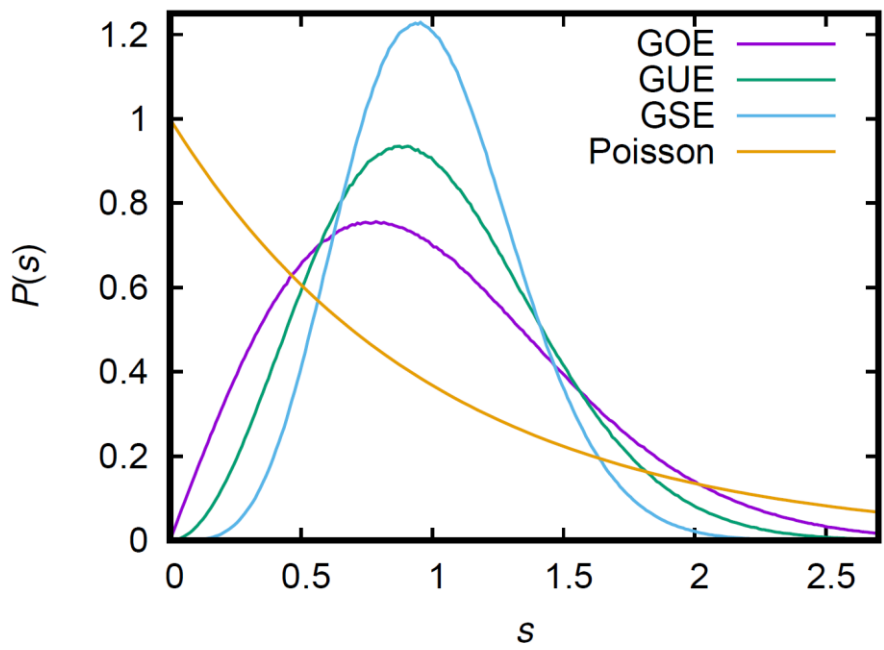


ガウス分布

Density $\propto e^{-\frac{\beta K}{4} \text{Tr} H^2} = \exp\left(-\frac{\beta K}{4} \sum_{i,j}^K |a_{ij}|^2\right)$

- 実 ($\beta=1$): Gaussian Orthogonal Ensemble (GOE)
- 複素 ($\beta=2$): G. Unitary E. (GUE)
- 四元数 ($\beta=4$): G. Symplectic E. (GSE)

- $N \equiv 0 \pmod{8}$
- $N \equiv 2, 6 \pmod{8}$
- $N \equiv 4 \pmod{8}$



固有値 $\{e_j\}$ の同時分布関数

準位反発

$$p(e_1, e_2, \dots, e_K) \propto \prod_{1 \leq i < j \leq K} |e_i - e_j|^\beta \prod_{i=1}^K e^{-\beta K e_i^2 / 4}$$

- $P(s)$: 正規化された準位間隔の分布 $s_j = \frac{e_{j+1} - e_j}{\Delta(\bar{e})}$
GOE/GUE/GSE: $P(s) \propto s^\beta$ で立ち上がり e^{-s^2} で減衰
相関なし: $P(s) = e^{-s}$ (ポアソン分布)
- $\langle r \rangle$: 隣接準位間隔比の平均

$$r = \frac{\min(e_{i+1} - e_i, e_{i+2} - e_{i+1})}{\max(e_{i+1} - e_i, e_{i+2} - e_{i+1})}$$

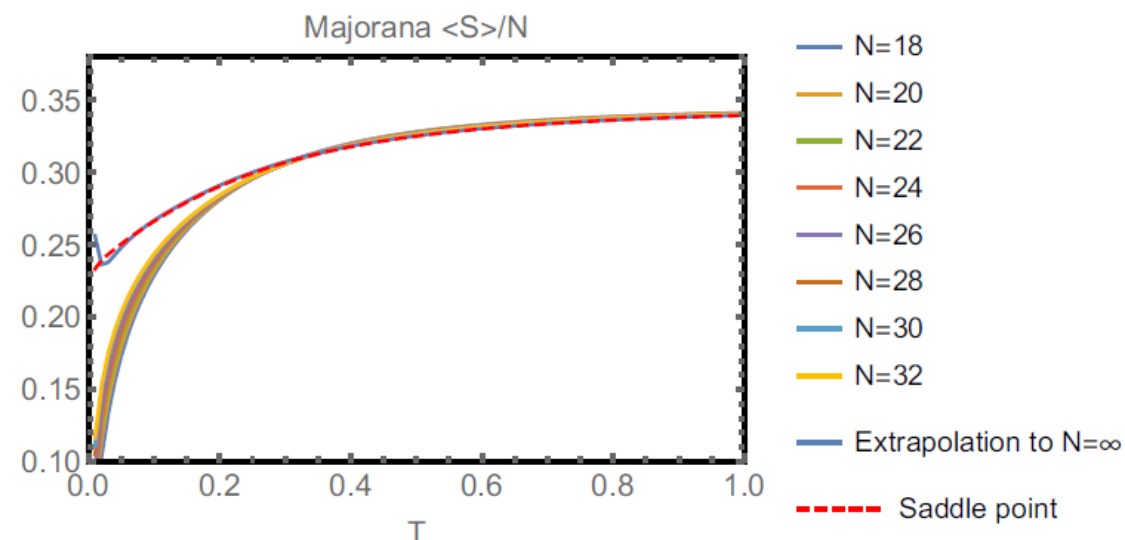
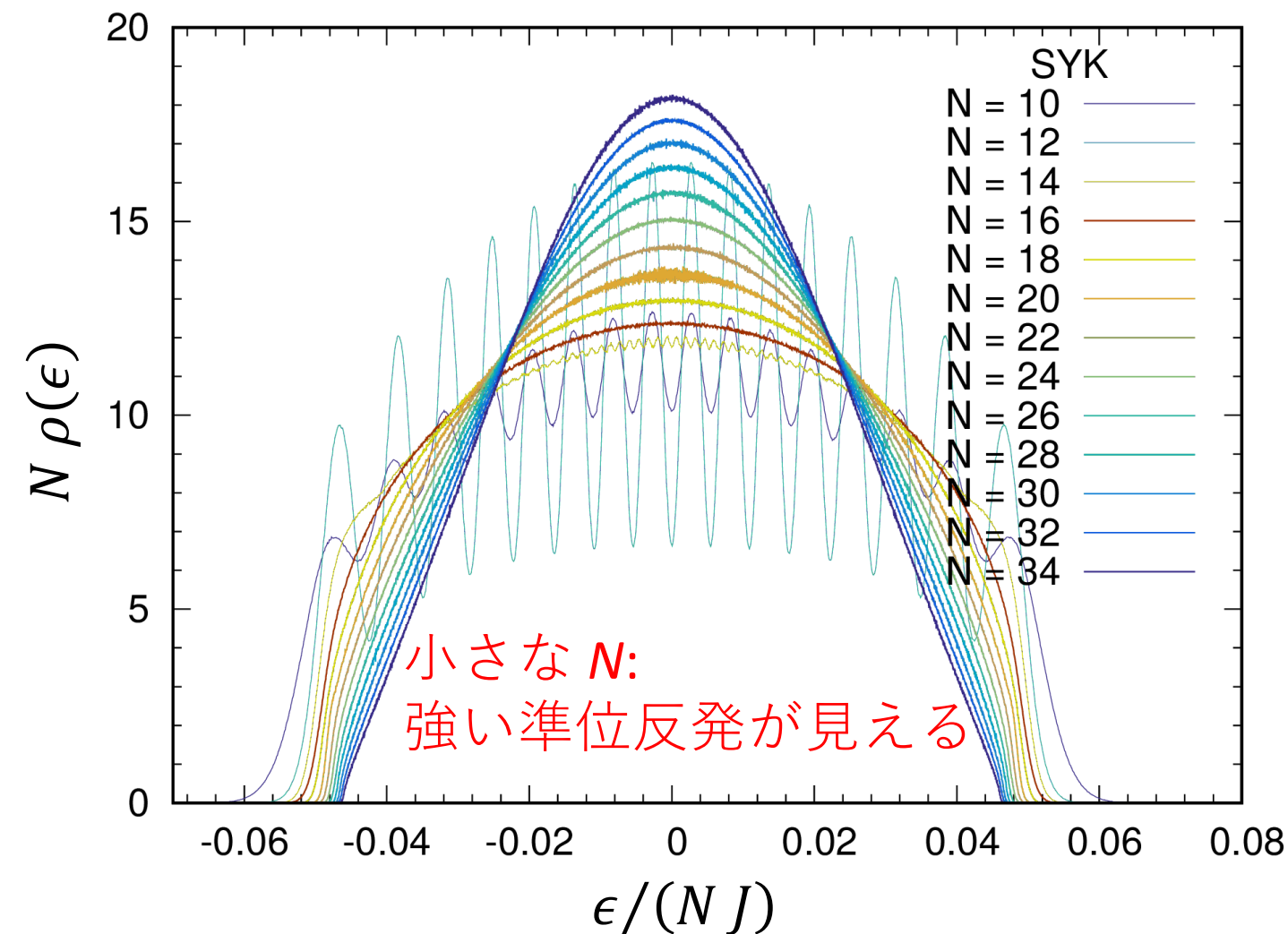
	Uncorrelated	GOE	GUE	GSE
$\langle r \rangle$	$2 \log 2 - 1 = 0.38629\dots$	0.5307(1)	0.5996(1)	0.6744(1)

[Y. Y. Atas et al. PRL 2013]

➔ SYK 模型: 準位相関は微視的には (同じ対称性の) ガウシアンアンサンブルと一致

ハミルトニアンの数値的対角化 → 固有値スペクトル

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \leq a < b < c < d \leq N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d \quad J_{abcd} : \text{Gaussian and variance } \sigma^2 = J^2$$



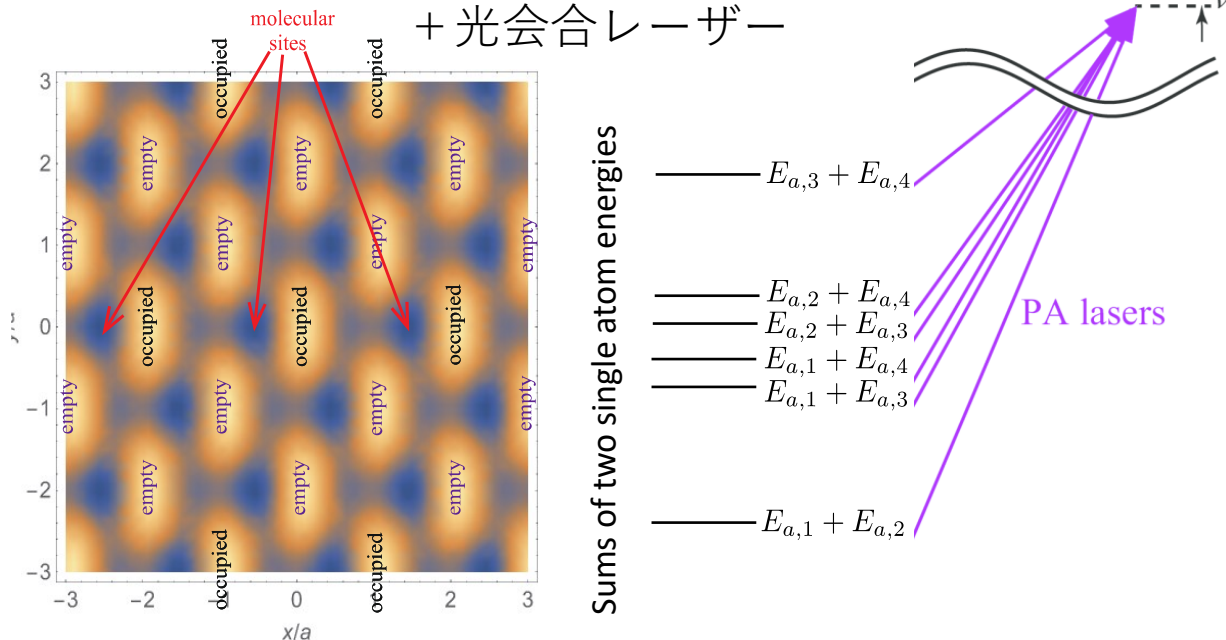
Large N への外挿:
低温で有限のエントロピーが残る

cf. Analytical spectral density for large N [A. M. García-García and J. J. M. Verbaarschot: PRD **96**, 066012 (2017)]

SYK模型の実験提案

[I. Danshita, M. Hanada, MT: PTEP **2017**, 083I01 (2017)]

光格子中の極低温フェルミ原子
+ 光会合レーザー



s : 分子のエネルギー準位のラベル

$$\hat{H}_m = \sum_{s=1}^{n_s} \left\{ \nu_s \hat{m}_s^\dagger \hat{m}_s + \sum_{i,j} g_{s,ij} \left(\hat{m}_s^\dagger \hat{c}_i \hat{c}_j - \hat{m}_s \hat{c}_i^\dagger \hat{c}_j^\dagger \right) \right\}.$$



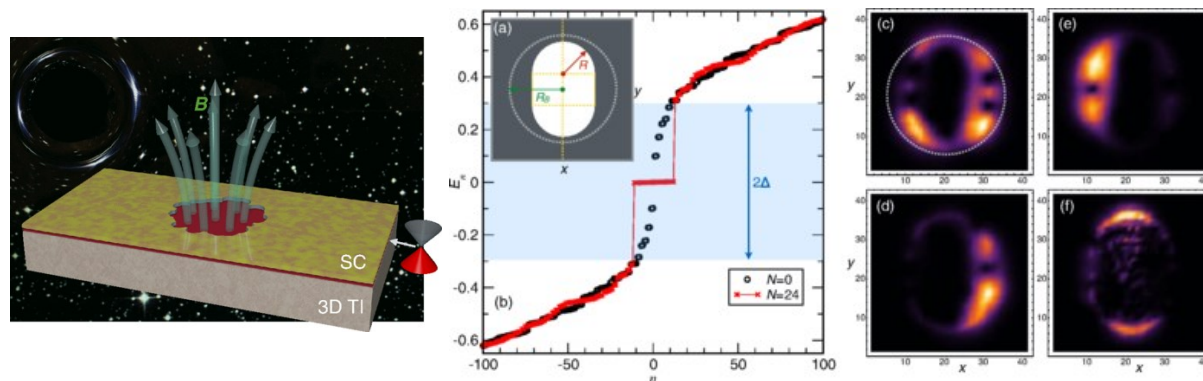
$$|\nu_s| \gg |g_{s,ij}|$$

$$\hat{H}_{\text{eff}} = \sum_{s,i,j,k,l} \frac{g_{s,ij} g_{s,kl}}{\nu_s} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l.$$

量子回路 [L. García-Álvarez et al., PRL 2017]

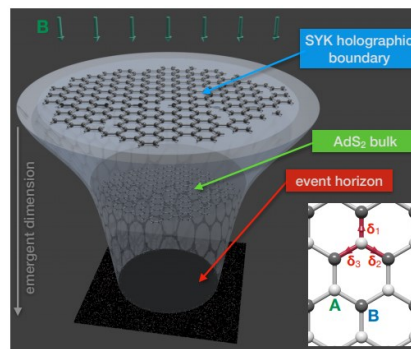
Majorana wire array [Chew, Essin, and Alicea, PRB 2017 (R)]

N 本の磁束量子に貫かれたトポロジカル超伝導体の小孔
[D. I. Pikulin and M. Franz, PRX **7**, 031006 (2017)]



磁場中のグラフェンの小片

[A. Chen et al., PRL **121**, 036403 (2018)]

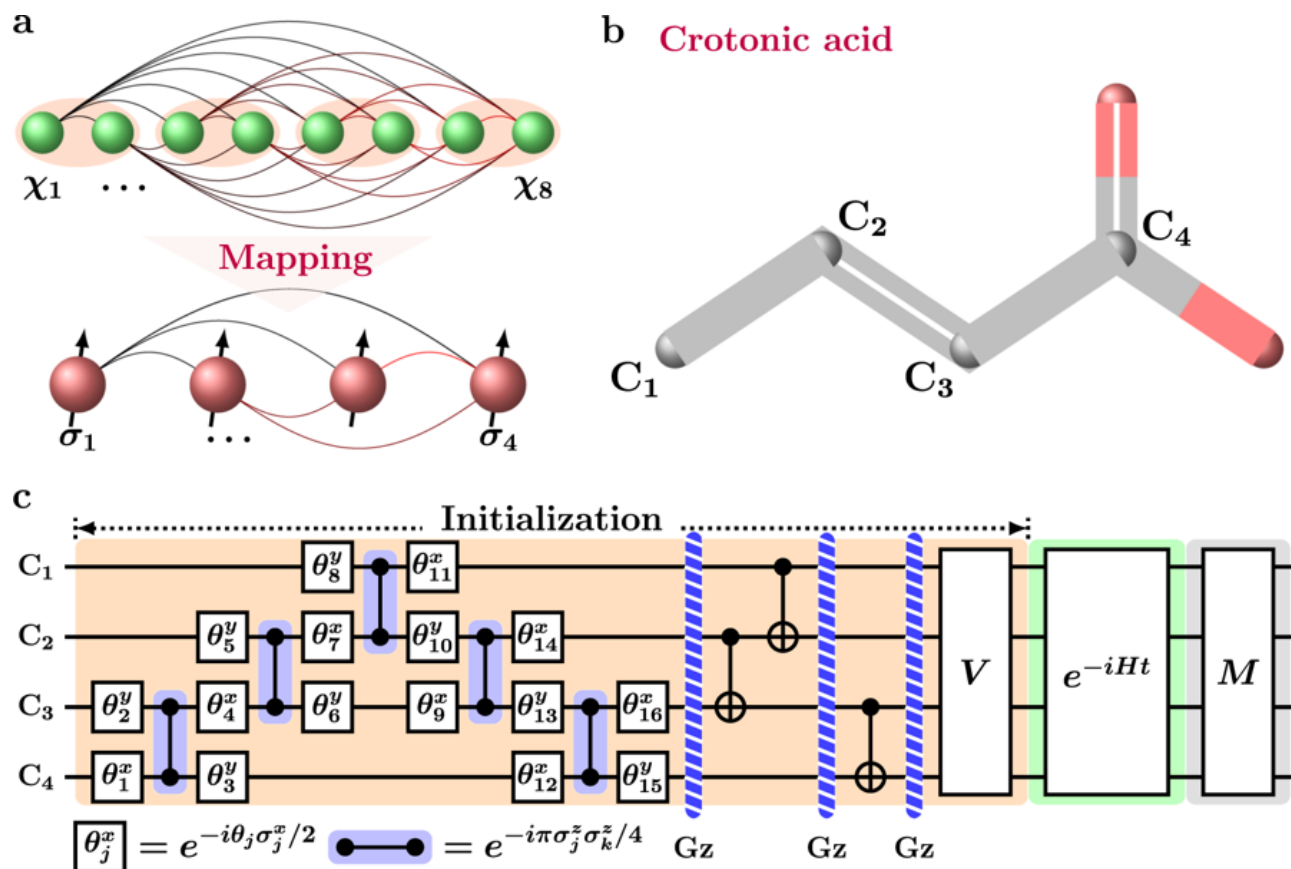


Review: M. Franz and M. Rozali,
“Mimicking black hole event horizons
in atomic and solid-state systems”,
Nature Reviews Materials **3**, 491 (2018)

NMR による SYK 模型の実現

“Quantum simulation of the non-fermi-liquid state of Sachdev-Ye-Kitaev model”

Zhihuang Luo, Yi-Zhuang You, Jun Li, Chao-Ming Jian, Dawei Lu, Cenke Xu, Bei Zeng and Raymond Laflamme, npj Quantum Information 5, 53 (2019)

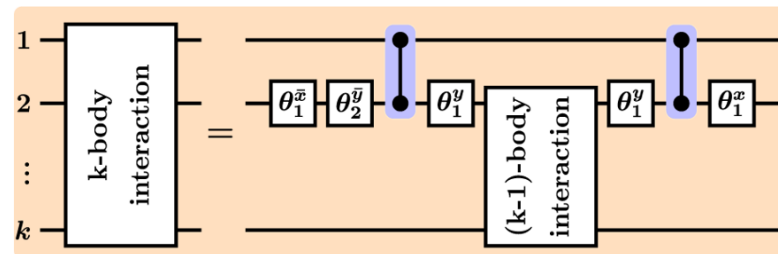


$$H = \frac{J_{ijkl}}{4!} \chi_i \chi_j \chi_k \chi_l + \frac{\mu}{4} C_{ij} C_{kl} \chi_i \chi_j \chi_k \chi_l$$

$$\chi_{2i-1} = \frac{1}{\sqrt{2}} \sigma_x^1 \sigma_x^2 \cdots \sigma_x^{i-1} \sigma_z^i, \chi_{2i} = \frac{1}{\sqrt{2}} \sigma_x^1 \sigma_x^2 \cdots \sigma_x^{i-1} \sigma_y^i.$$

$$H = \sum_{s=1}^{70} H_s = \sum_{s=1}^{70} a_{ijkl}^s \sigma_{\alpha_i}^1 \sigma_{\alpha_j}^2 \sigma_{\alpha_k}^3 \sigma_{\alpha_l}^4$$

$$e^{-iH\tau} = \left(\prod_{s=1}^{70} e^{-iH_s \tau/n} \right)^n + \sum_{s < s'} \frac{[H_s, H_{s'}] \tau^2}{2n} + O(|a|^3 \tau^3/n^2),$$



A. M. García-García, A. Romero-Bermúdez, B. Loureiro,
and MT, Phys. Rev. Lett. **120**, 241603 (2018)
also see: reply (arXiv:2007.06121) in press to comment (J. Kim and
X. Cao, arXiv:2004.05313).

SYK₄₊₂

Q.: カオスのダイナミクスのための最低条件？ (→ 重力側での解釈？)
なるべく単純なモデルを解析的・数値的手法で調べる

$$\hat{H} = \sum_{1 \leq a < b < c < d}^N \text{SYK}_4 J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d + i \sum_{1 \leq a < b}^N \text{SYK}_2 K_{ab} \hat{\chi}_a \hat{\chi}_b$$

ガウシアンランダム結合 J_{abcd} : 平均 0, 標準偏差 $\frac{\sqrt{6}J}{N^{3/2}}$ $J = 1$: unit of energy
 K_{ab} : 平均 0, 標準偏差 $\frac{K}{\sqrt{N}}$

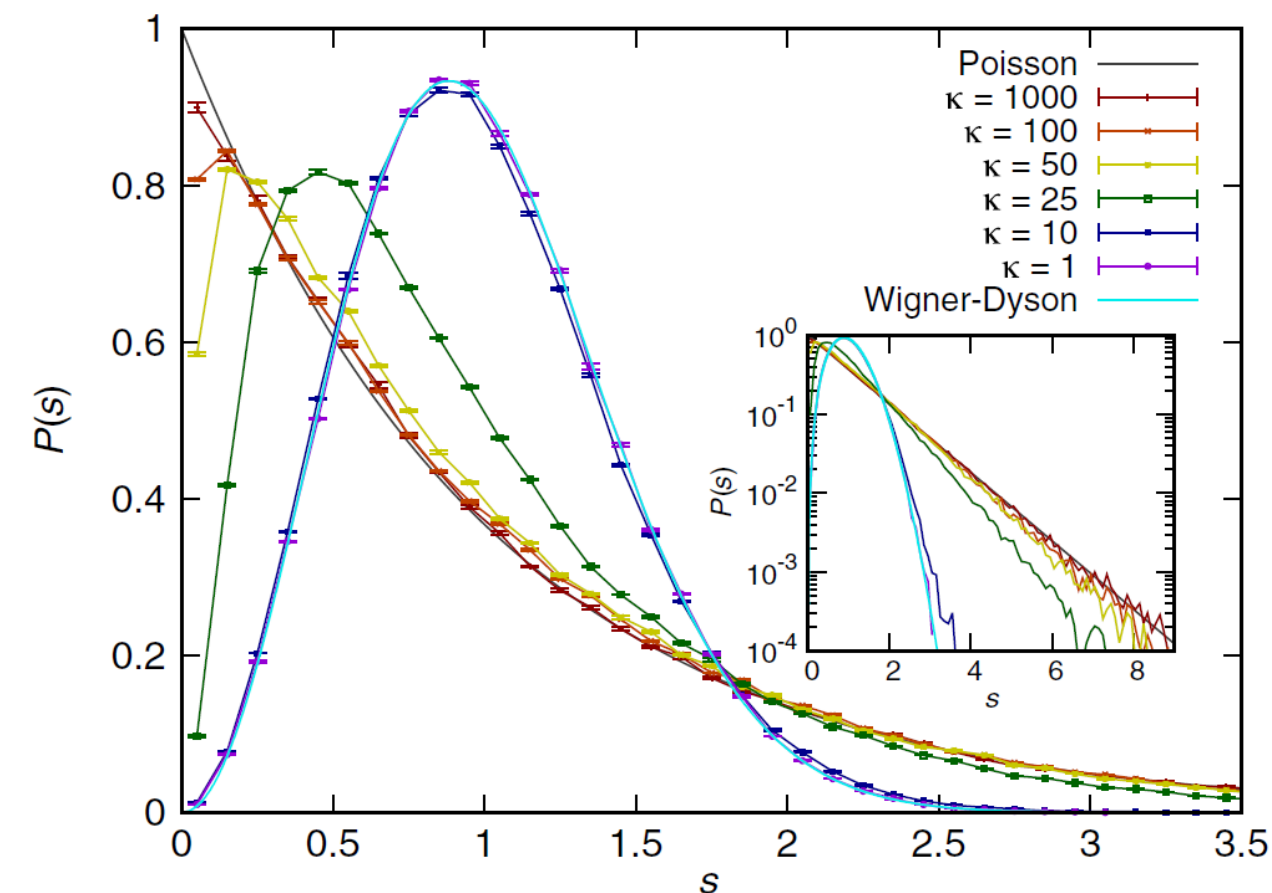
SYK₄ as unperturbed Hamiltonian,

K controls the strength of SYK₂ (one-body random term, solvable)

ここでは GUE
 $N \equiv 2 \pmod{4}$
に注目する。

どちらの項も、複素フェルミオンで書いたときのパリティを保存
→ $2^{N/2-1}$ 次元のハミルトニアン行列の完全数値対角化は $N \lesssim 34$ で可能

RMT-like behavior lost as SYK2 term is introduced



$P(s)$: level spacing distribution

Ratio of consecutive level spacing $E_{i+1} - E_i$
to the local mean level spacing Δ
(requires unfolding of the spectrum)

SYK₄ limit (small K):

Obeys random matrix theory (RMT)

(GUE (Gaussian Unitary Ensemble) if $N \equiv 2 \pmod{4}$)

SYK₂ (large K): Poisson (e^{-s})

$N=30$, Central 10 % of eigenvalues

Also see: T. Nosaka, D. Rosa, and J. Yoon, JHEP **1809**, 041 (2018) for other symmetry cases

cf. A. V. Lunin, K. S. Tikhonov, and M. V. Feigel'man, PRL **121**, 236601 (2018); Y. Yu-Xiang, F. Sun, J. Ye, and W. M. Liu, 1809.07577, ...

SYK_{q≥4} + SYK₂ : breakdown of chaos

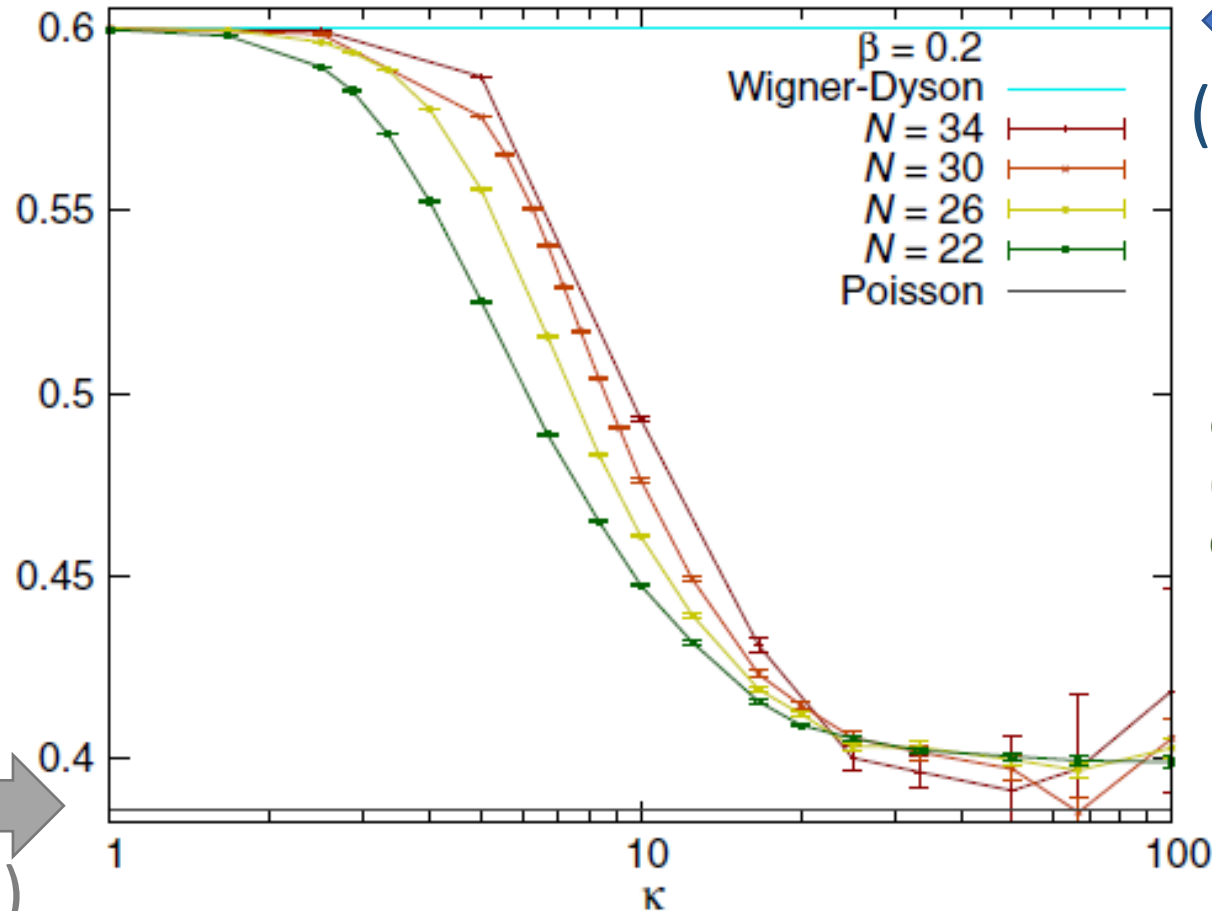
$$\hat{H} = \sum_{1 \leq a < b < c < d}^N J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d + i \sum_{1 \leq a < b}^N K_{ab} \hat{\chi}_a \hat{\chi}_b$$

K_{ab} : standard deviation = $\kappa / \sqrt{N}$

Averaged
ratio between
neighboring
energy level
separations

 $\langle r \rangle_\beta$

Poisson
(uncorrelated)



← GUE
(Gaussian Unitary Ensemble)

Lyapunov exponent calculated in the large- N limit: also deviates from the chaos bound, approaches zero at low T (see also our reply 2007.06121 to a comment, PRL in press)

We consider N Majorana fermions with normalization $\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$ here

Deviation from Gaussian random matrix as SYK₂ component is introduced

Many-body localization

ETH: “(almost) all eigenstates are thermal
(expectation values of operators = microcanonical average)”

- Anderson localization: concept in non-interacting systems
 - Localization of wavefunctions due to scatterings at impurities
 - Many experiments in cold atom gases, optical fibers, etc.
- MBL: does localization occur in interacting systems?

[Gornyi, Mirlin, Polyakov 2005, Basko, Aleiner, Altshuler 2006, Oganesyan and Huse 2007, ... many others]

- Memory of initial conditions remains accessible at long times
- Reduced density matrix on a subsystem does not approach a thermal one
- Energy eigenstates do not obey Eigenstate Thermalization Hypothesis (ETH)
- Area law, rather than volume law, of entanglement entropy
- “Standard model”: spin-1/2 Heisenberg model + random field in z direction
 - Much debate on the location of the localization transition

$$\hat{H} = \sum_i^N \hat{S}_i \cdot \hat{S}_{i+1} + \sum_i^N h_i \hat{S}_i^z$$

$h_i \in [-h, h]$ uniform distribution

Our model and choice of basis

$$\text{SYK}_4 + \delta \text{SYK}_2$$

$$\hat{H} = - \sum_{1 \leq a < b < c < d}^{N=2N_D} J'_{abcd} \hat{\psi}_a \hat{\psi}_b \hat{\psi}_c \hat{\psi}_d + i \sum_{1 \leq a < b}^N K_{ab} \hat{\psi}_a \hat{\psi}_b$$

Block-diagonalize the SYK₂ part
(the skew-symmetric matrix (K_{ab}) has eigenvalues $\pm v_j$)

$$\hat{H} = - \sum_{1 \leq a < b < c < d}^{2N_D} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d + i \sum_{1 \leq j \leq N}^{2N_D} v_j \hat{\chi}_{2j-1} \hat{\chi}_{2j}$$

Normalization of J_{abcd} , v_j :
SYK₄ bandwidth = 1,
Width of v_j distribution = δ

We choose $\{\hat{\psi}_a, \hat{\psi}_b\} = \{\hat{\chi}_a, \hat{\chi}_b\} = 2\delta_{ab}$ as the normalization for the $N = 2N_D$ Majorana fermions.
For $\hat{c}_j = \frac{1}{2}(\hat{\chi}_{2j-1} + i\hat{\chi}_{2j})$ we have $\{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{ij}$.

Our model and choice of basis

$N = 2N_D = 14: 2^7 = 128$ states

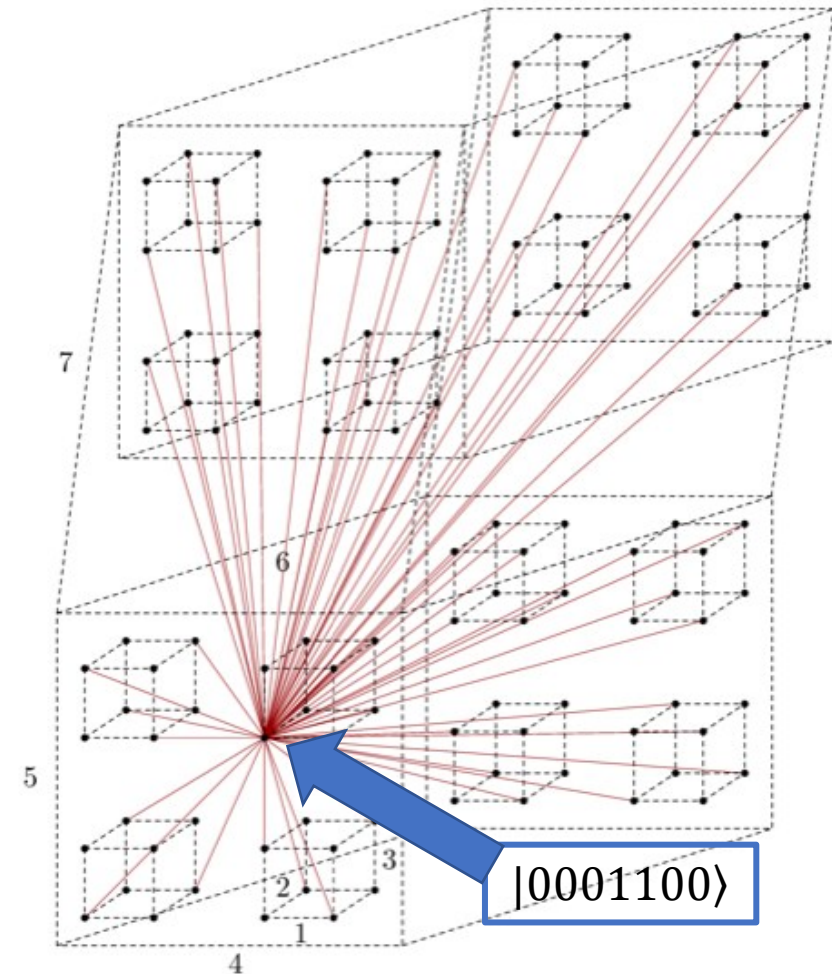
Basis diagonalizing the complex fermion number operators
 $\hat{n}_j = \hat{c}_j^\dagger \hat{c}_j \rightarrow$ Sites: the 2^{N_D} vertices of an N_D -dim. hypercube.

$$\hat{c}_j = \frac{1}{2}(\hat{\chi}_{2j-1} + i\hat{\chi}_{2j})$$

$$\begin{aligned} \hat{H} &= - \sum_{1 \leq a < b < c < d}^{2N_D} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d + i \sum_{1 \leq j \leq N}^{N_D} v_j \hat{\chi}_{2j-1} \hat{\chi}_{2j} \\ &= - \sum_{1 \leq a < b < c < d}^{2N_D} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d + \sum_{1 \leq j \leq N}^{N_D} v_j (2\hat{n}_j - 1) \end{aligned}$$

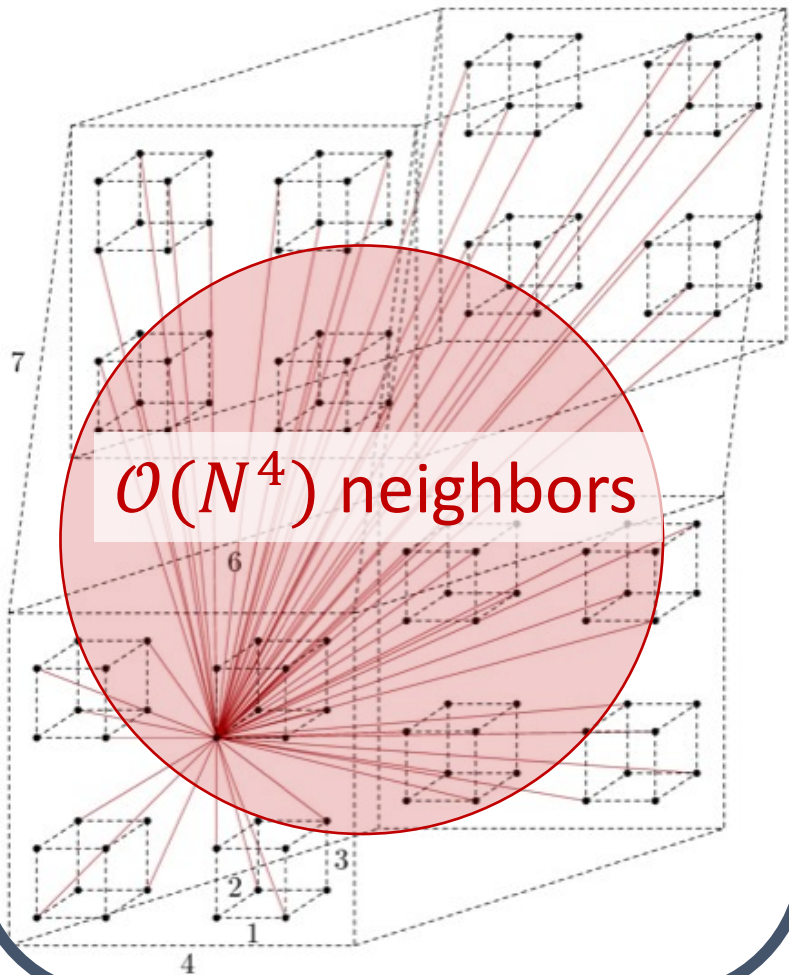
Each term of SYK₄ connects vertices with distance = 0, 2, 4.

For $N = 14$, each vertex is directly connected with
 1 (distance=0, itself) + 21 (distance=2) + 35 (distance=4)
 vertices out of the possible $2^N = 128$ (64 per parity).



Our model and choice of basis

2^{N_D} Fock states



Basis diagonalizing the complex fermion number operators $\hat{n}_j = \hat{c}_j^\dagger \hat{c}_j \rightarrow$ Sites: the 2^{N_D} vertices of an N_D -dim. hypercube.

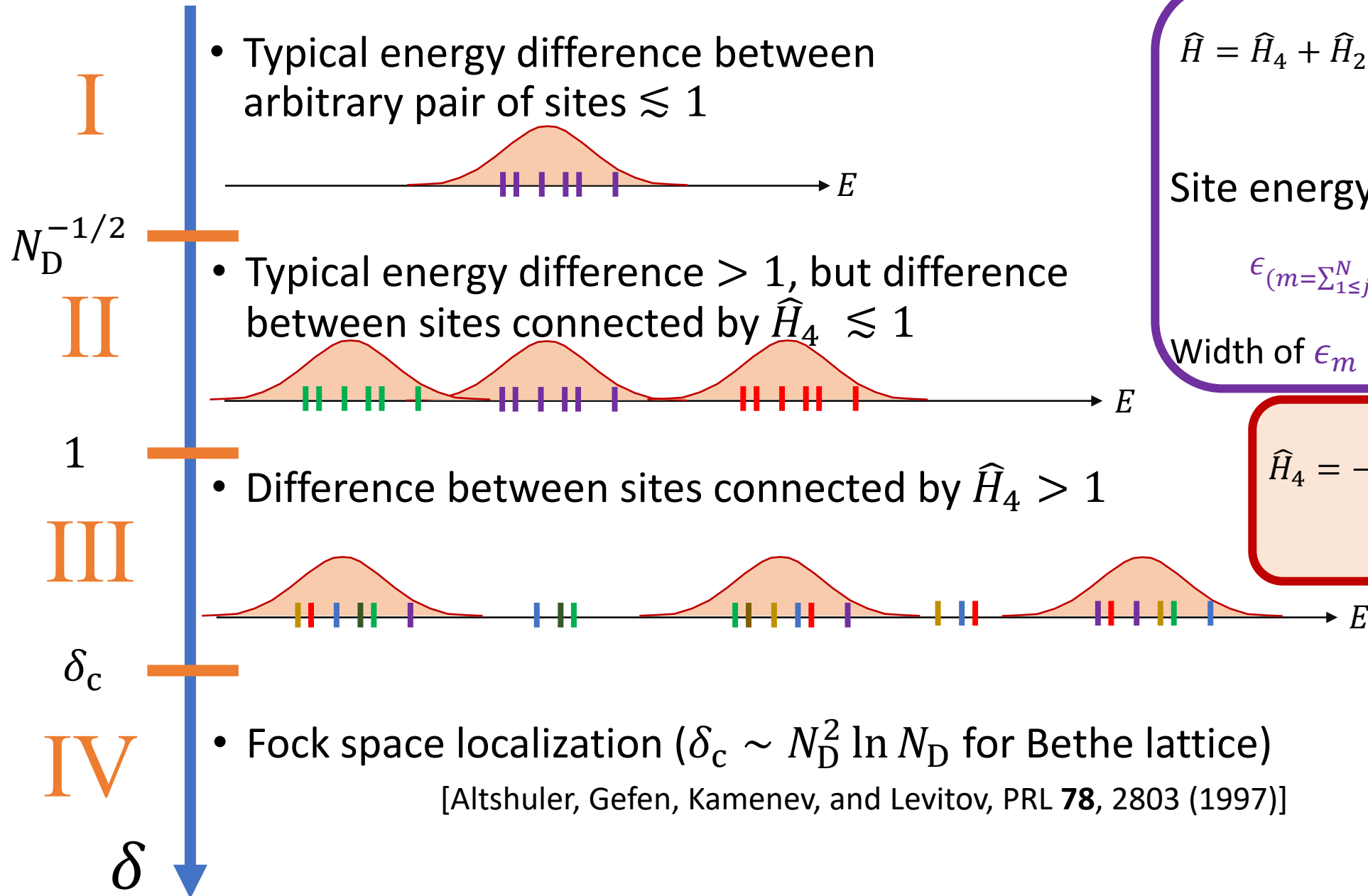
$$\text{SYK}_4 + \delta \text{SYK}_2$$

$$\hat{H} = - \sum_{1 \leq a < b < c < d}^{2N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d + \sum_{1 \leq j \leq N}^N v_j (2\hat{n}_j - 1)$$

Each term of SYK_4 connects vertices with distance = 0, 2, 4.

For $N = 34$, each vertex is directly connected with **1 (distance=0, itself) + 136 (distance=2) + 2380 (distance=4) vertices** out of the possible $2^{N/2} = 131072$ (65536 per parity).

Four regimes of disorder strengths



$$\hat{H} = \hat{H}_4 + \hat{H}_2 \quad \hat{H}_2 = \sum_{1 \leq j \leq N} v_j (2\hat{n}_j - 1)$$

width of v_j dist. = δ

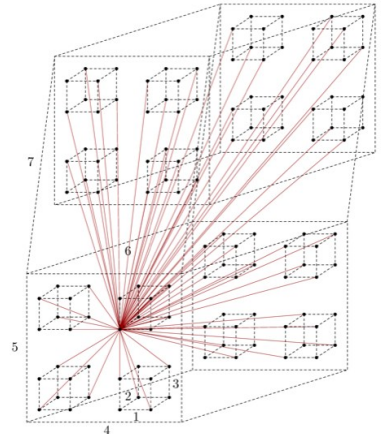
Site energy of site # m :

$$\epsilon_{(m=\sum_{1 \leq j \leq N} 2^{j-1} n_j)} = \sum_{1 \leq j \leq N} (-1)^{n_j-1} v_j$$

Width of ϵ_m dist. = $\sqrt{N_D} \delta$

$$\hat{H}_4 = - \sum_{1 \leq a < b < c < d}^{2N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

SYK₄ bandwidth = 1



Diagnostic quantities: Moments of wave functions and spectral two-point correlation function

- Moments of eigenstate wave functions

$$I_q = \nu^{-1} \sum_{n,\psi} \langle |\langle \psi | n \rangle|^{2q} \delta(E_\psi) \rangle_J$$

with average density of states at band center

$$\nu = \nu(E \simeq 0), \nu(E) = \sum_{\psi} \langle \delta(E - E_\psi) \rangle_J$$

➔ Parametrizes localization, allows comparison with numerics

$$I_2 = \nu^{-1} \sum_{n,\psi} \langle |\langle \psi | n \rangle|^4 \delta(E_\psi) \rangle_J:$$

inverse participation ratio (IPR), $\frac{1}{D} \leq I_2 \leq 1$

Equal weights

Single non-zero element

D : dimension of $\{|n\rangle\} = 2^{N-1}$

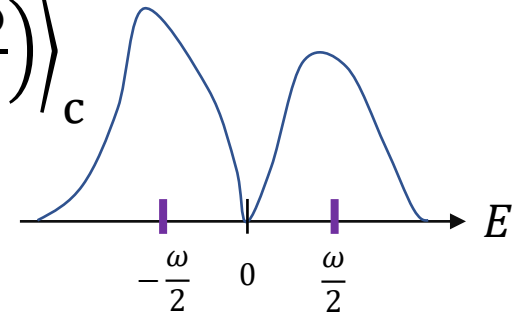
- Spectral two-point correlation function

$$K(\omega) = \nu^{-2} \left\langle \nu\left(\frac{\omega}{2}\right) \nu\left(-\frac{\omega}{2}\right) \right\rangle_c$$

c : connected part

$$\langle AB \rangle_c = \langle AB \rangle_J - \langle A \rangle_J \langle B \rangle_J$$

➔ Reflects level repulsion if the spectrum is random matrix-like



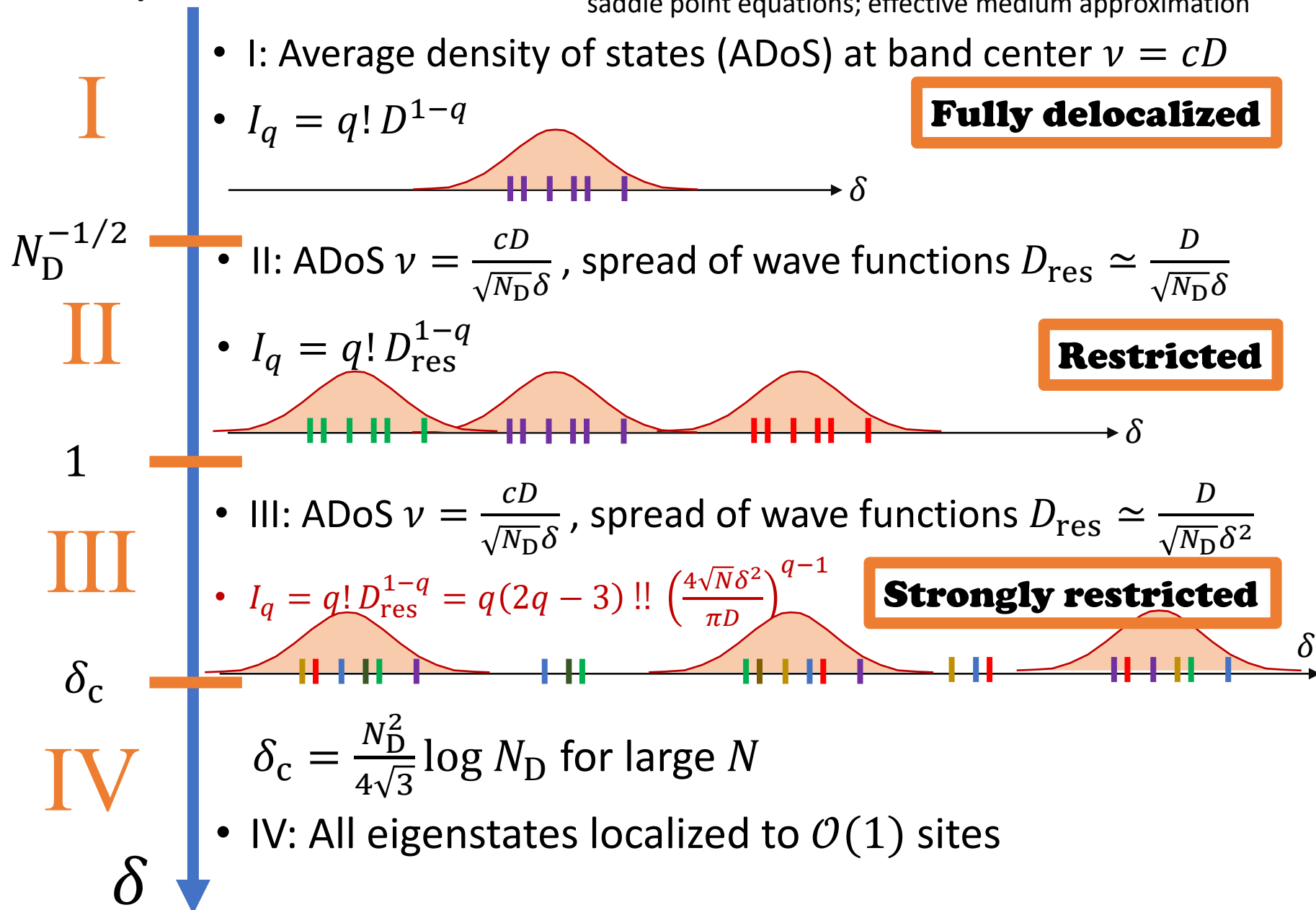
➔ We calculate these quantities for large N and compare against numerical results

Analytical results

Method: Exact matrix integral representation of I_q and $K(\omega)$;
mapping to a supersymmetric sigma model;
saddle point equations; effective medium approximation

PRR 3, 013023 (2021)

$$(N_D = \frac{N}{2}, c = O(1), D = 2^{N_D-1})$$



Eigenenergy spectral statistics (for odd N case for simplicity)

$$\tilde{K}(s) = 1 - \frac{\sin^2 s}{s^2} + \delta \left(\frac{s}{\pi} \right),$$

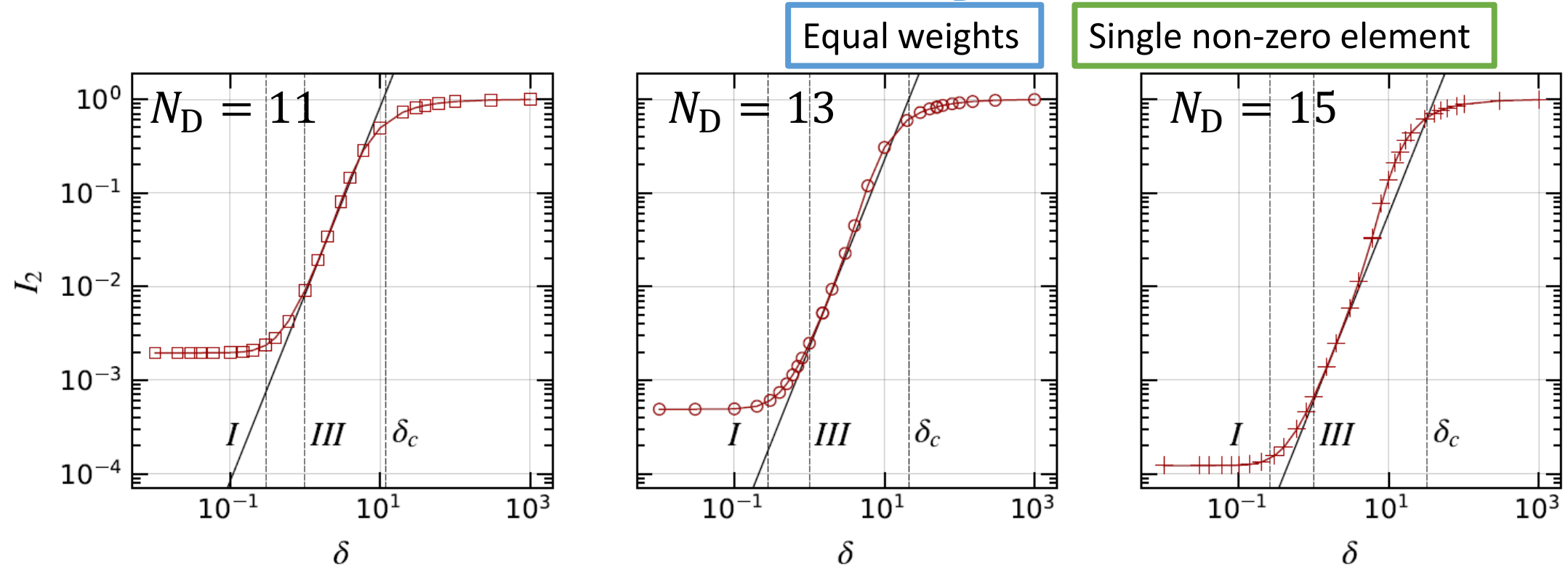
$s = \pi\omega\nu$ in I, II, III :

agrees with Gaussian Unitary Ensemble (GUE)

IV: Poisson statistics

Inverse participation ratio vs prediction for III

IPR I_2 = average of $\sum_n |\langle \psi | n \rangle|^4$ for normalized ψ , $\frac{1}{D} \leq I_2 \leq 1$



$$I_q = \frac{q(2q-3)!!}{\delta^{2(1-q)}} \left(\frac{\pi D}{4\sqrt{N_D}} \right)^{1-q} = q(2q-3)!! \left(\frac{4\sqrt{N_D}\delta^2}{2^{N-1}\pi} \right)^{q-1} \text{ in III}$$

Central 1/7 of the energy spectrum

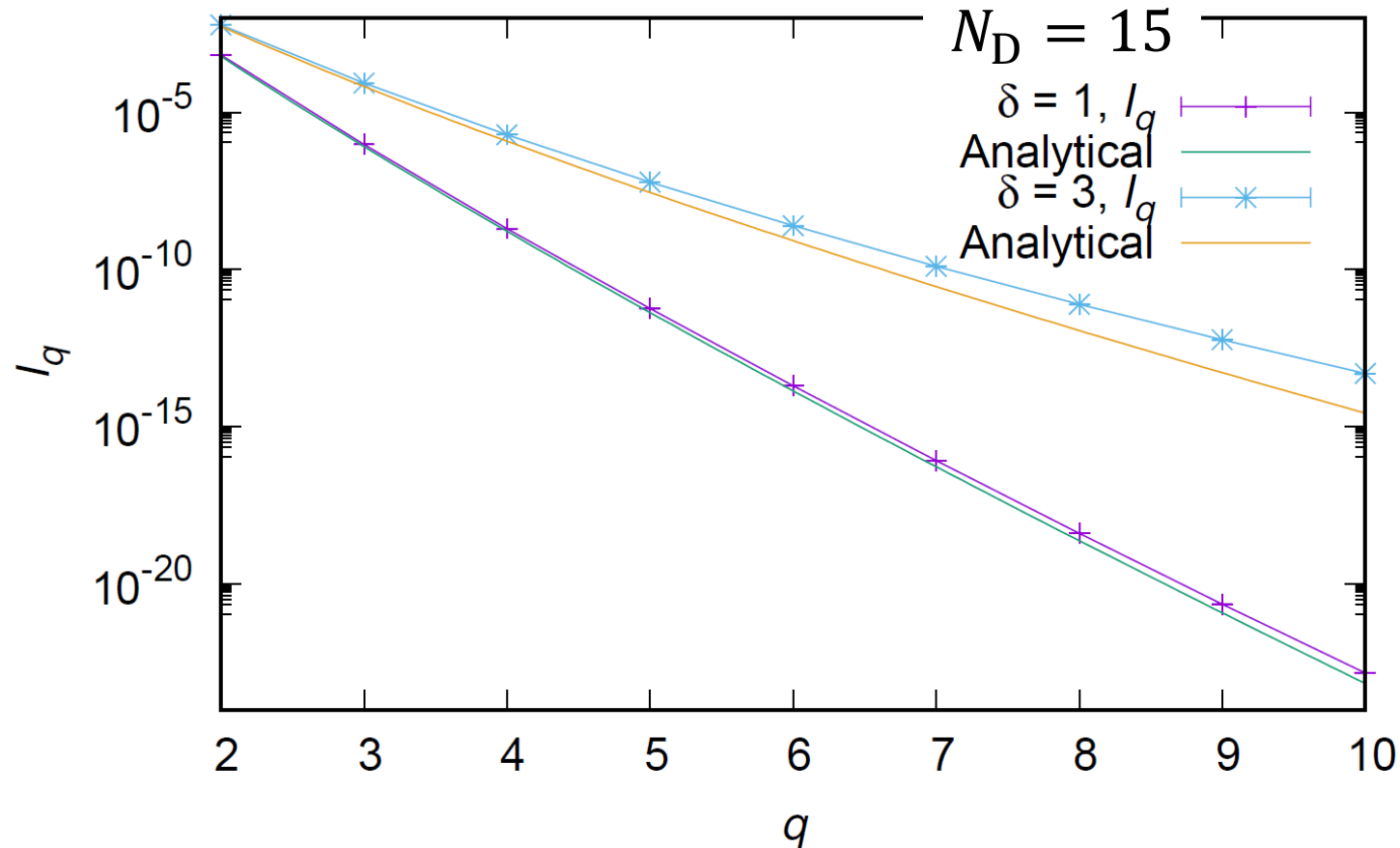
Higher moments of eigenvectors

PRR 3, 013023 (2021)

Analytical prediction:

$$I_q = \frac{q(2q-3)!!}{\delta^{2(1-q)}} \left(\frac{\pi D}{4\sqrt{N_D}} \right)^{1-q} = q(2q-3)!! \left(\frac{4\sqrt{N_D}\delta^2}{2^{N-1}\pi} \right)^{q-1} \text{ in III}$$

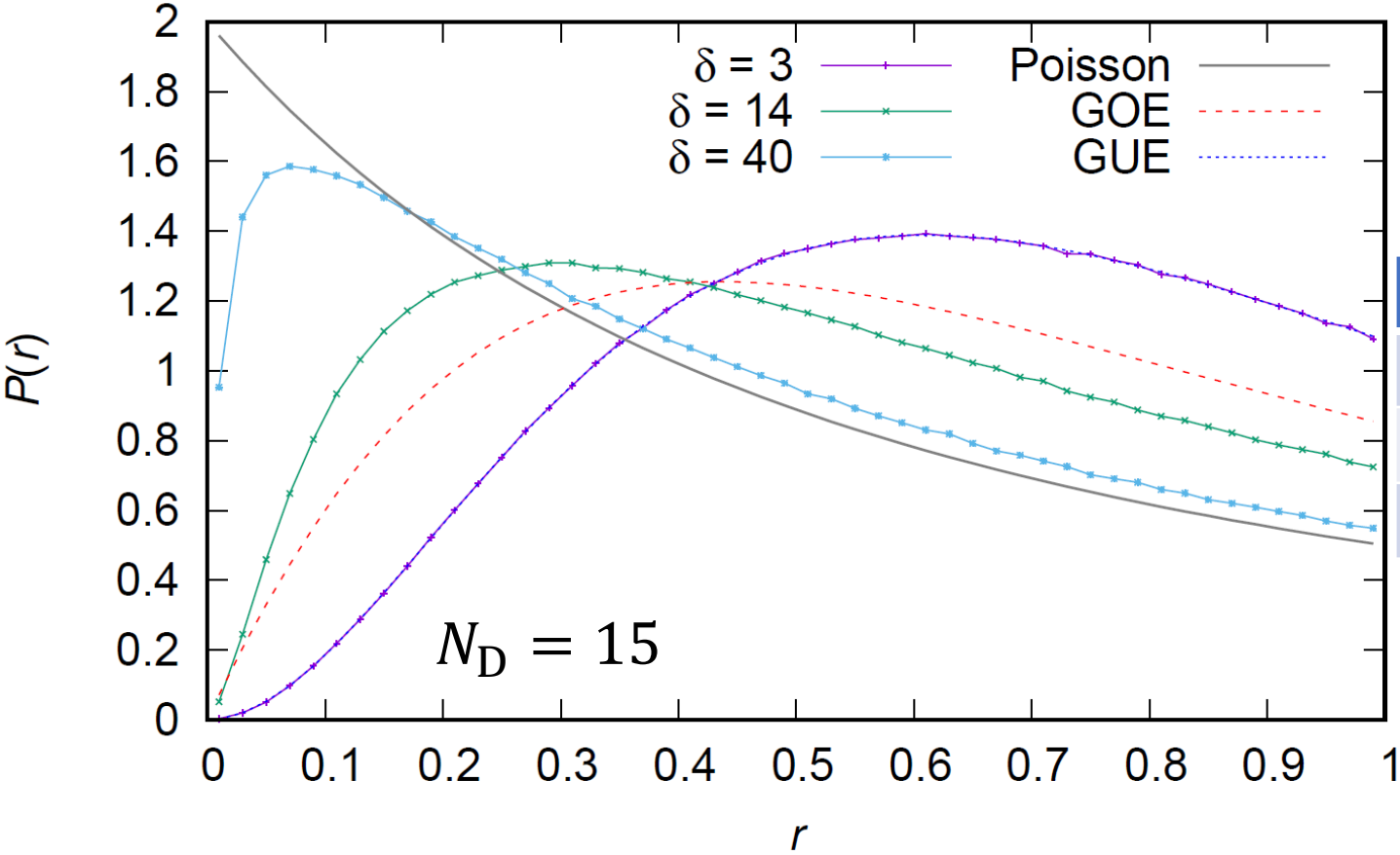
$$I_q = v^{-1} \sum_{n,\psi} \langle |\langle \psi | n \rangle|^{2q} \delta(E_\psi) \rangle_J$$



Good agreement up to large q for $\delta \sim 1$

Central 1/7 of the energy spectrum

Spectral statistics: gap ratio distribution



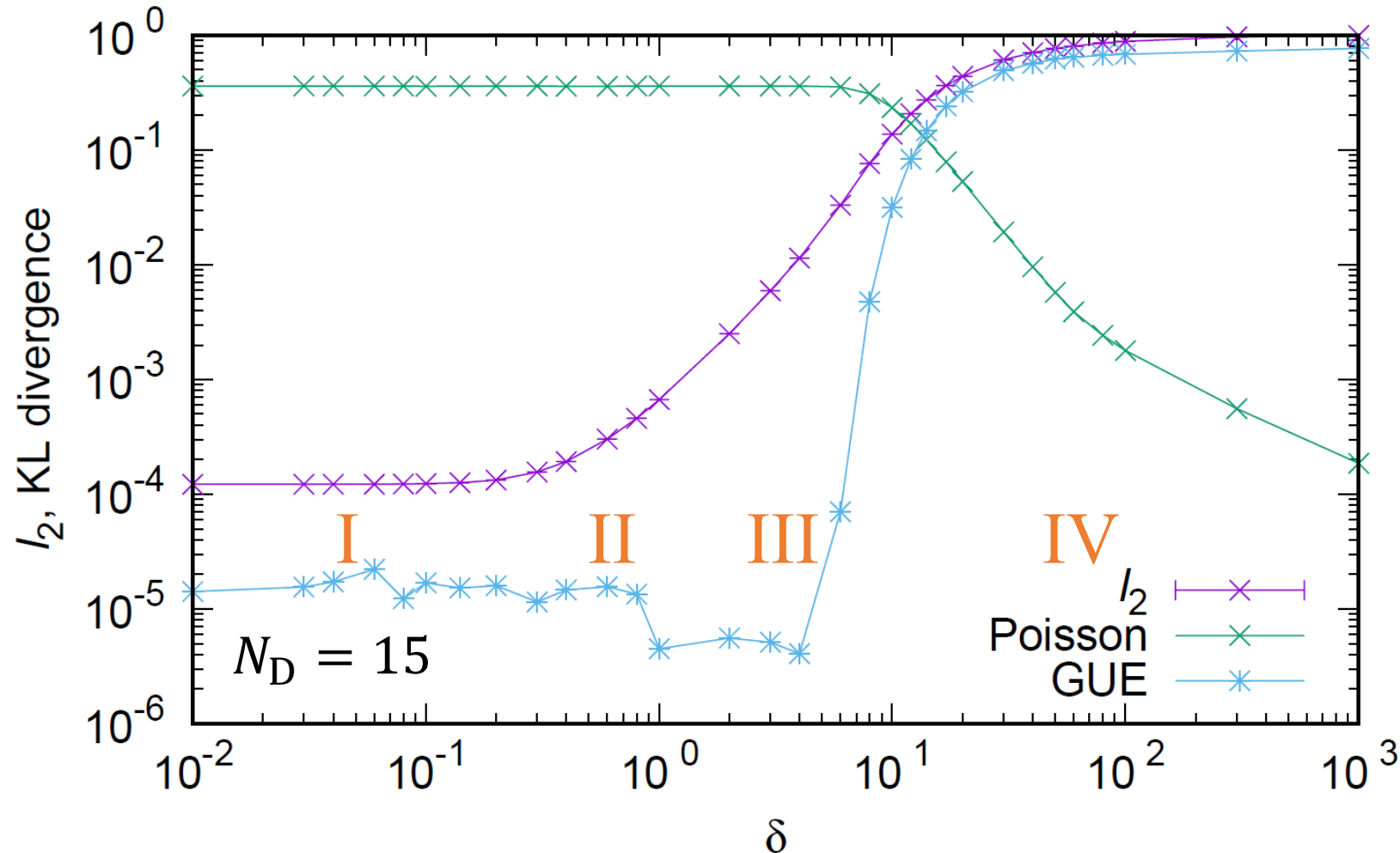
Measure difference by Kullback-Leibler (KL) divergence: $D_{\text{KL}}(P||Q) = \sum_x P(x) \log \frac{P(x)}{Q(x)}$.

δ	$D_{\text{KL}}(P(\delta, r) P_{\text{Poisson}}(r))$	$D_{\text{KL}}(P(\delta, r) P_{\text{GUE}}(r))$
3	0.3608	5×10^{-6}
14	0.1234	0.1463
40	0.0096	0.5705

$$r = \frac{\min(E_{i+1} - E_i, E_{i+2} - E_{i+1})}{\max(E_{i+1} - E_i, E_{i+2} - E_{i+1})}$$

$$(\delta_c = \frac{Z}{\sqrt{2\rho}} W(2Z\sqrt{\pi}) = 38.47)$$

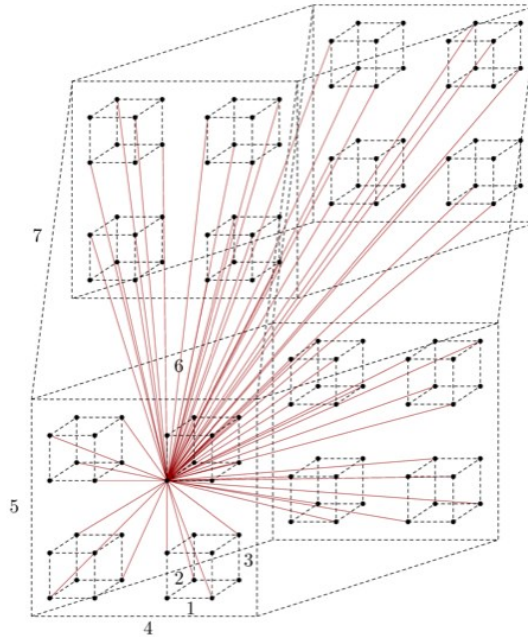
Departure from random matrix $P(r)$ occurs after I_2 has grown significantly



Summary so far...

Fock space localization in many-body quantum systems

Analytical estimate of inverse participation ratio, spectral statistics



Sachdev-Ye-Kitaev model as tractable system

Numerical calculation of inverse participation ratio, energy spectrum correlation

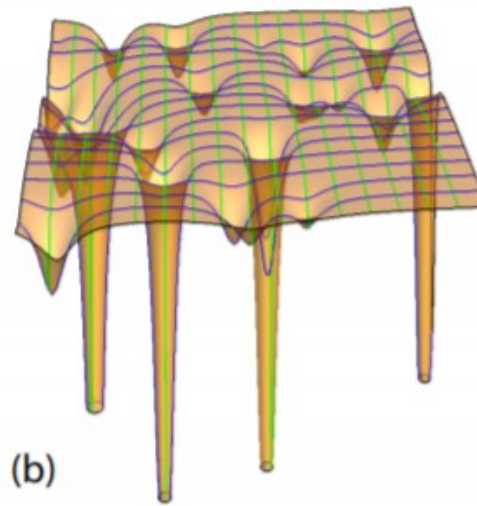
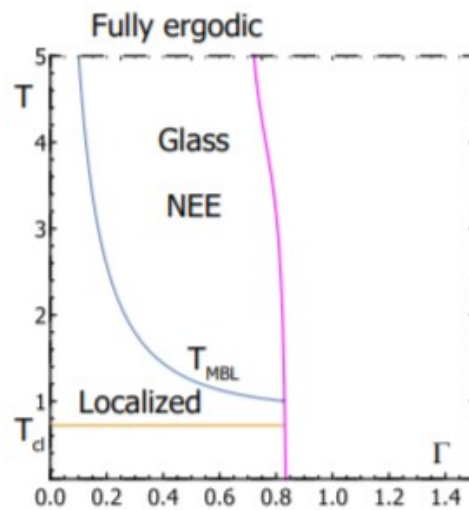
Four regimes (I: ergodic, II: localization starts, III: localization rapidly progresses, IV: MBL) found in $\text{SYK}_4 + \delta \text{SYK}_2$ system (in SYK_2 -diagonal basis); I, II, III are chaotic while IV is not

Prediction for momenta of eigenstate wavefunctions I_q is verified by **parameter free comparison**, and **energy spectrum statistics** is consistent with GUE/Poisson transition well after entering regime III

➔ Behavior of the entanglement entropy?

Physics just outside MBL (regions II & III)?

- Thermal phase smoothly connected to extended states (as those in translationally invariant models)?
- Non-ergodic extended (NEE) states discussed for several models (Bethe lattice, random regular graphs, disordered Josephson junction chains, ...)

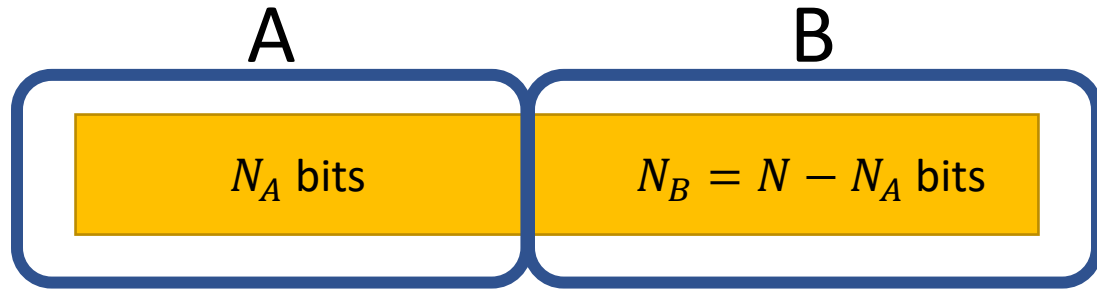


“golf course” potential energy landscape

“Non-ergodic extended phase of the Quantum Random Energy Model”

[L. Faoro, M. V. Feigel'man, L. Ioffe, Ann. Phys. **409**, 167916 (2019)]

Evaluation of entanglement entropy



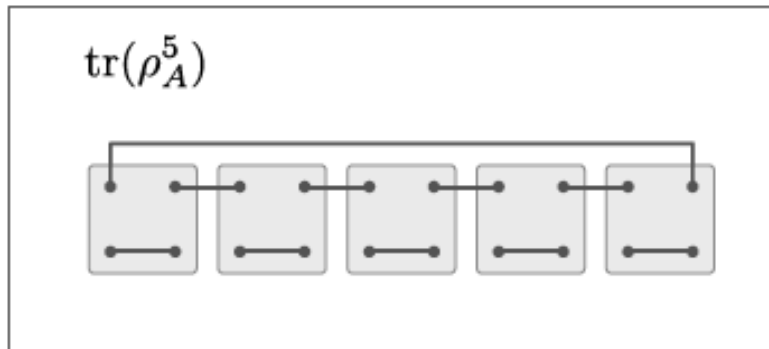
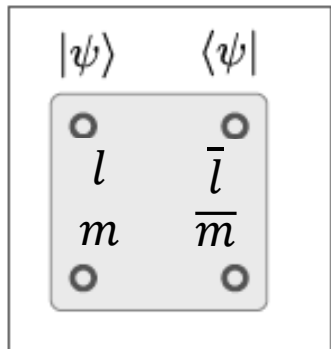
Zero-energy eigenstate $|\psi\rangle$, density matrix $\rho = |\psi\rangle\langle\psi|$

Reduced density matrix $\rho_A = \text{tr}_B \rho$

Entanglement entropy $S_A = -\text{tr}_A(\rho_A \ln \rho_A)$

Fock space $\mathcal{F} = \mathcal{F}_A \otimes \mathcal{F}_B$
 $n = (l, m)$

Evaluate disorder averaged moments $M_r = \langle \text{tr}_A(\rho_A^r) \rangle$, $S_A = -\partial_r M_r|_{r=1}$.



$\mathcal{N} = (n^1, n^2, \dots, n^r)$, $\mathcal{N}_A = (l^1, l^2, \dots, l^r)$, $\mathcal{N}_B = (m^1, m^2, \dots, m^r)$

$$\rho_A^r = \sum_{\substack{l^1, \dots, l^r \\ m^1, \dots, m^r}} \psi^{(l^1, m^1)} \bar{\psi}^{(l^2, m^1)} \psi^{(l^2, m^2)} \bar{\psi}^{(l^3, m^2)} \dots \psi^{(l^r, m^r)} \bar{\psi}^{(l^1, m^r)}$$

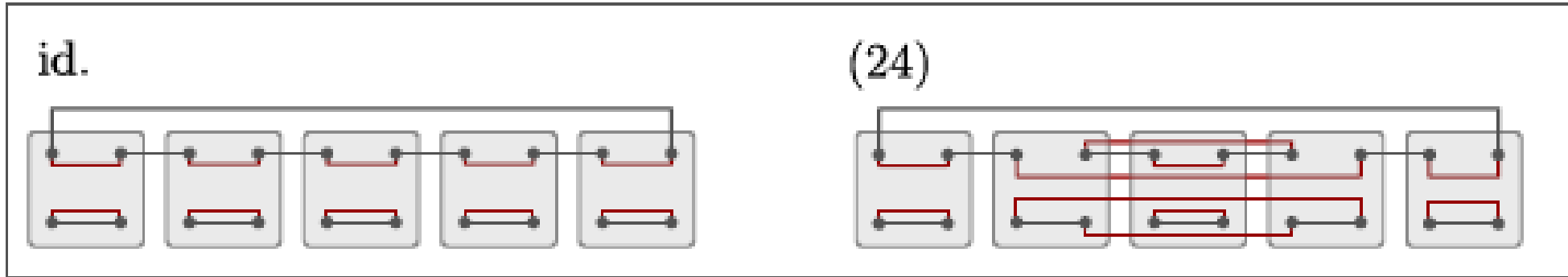
Evaluation of power of reduced density matrix

$$\rho_A^r = \sum_{\substack{l^1, \dots, l^r \\ m^1, \dots, m^r}} \psi^{(l^1, m^1)} \bar{\psi}^{(l^2, m^1)} \psi^{(l^2, m^2)} \bar{\psi}^{(l^3, m^2)} \dots \psi^{(l^r, m^r)} \bar{\psi}^{(l^1, m^r)}$$

For this sum to survive disorder averaging,

$\mathcal{N} = (n^1, n^2, \dots, n^r)$ and $\bar{\mathcal{N}} = (\bar{n}^1, \bar{n}^2, \dots, \bar{n}^r)$ should be equal as sets,

$$\mathcal{N}^i = \bar{\mathcal{N}}^{\sigma(i)}$$

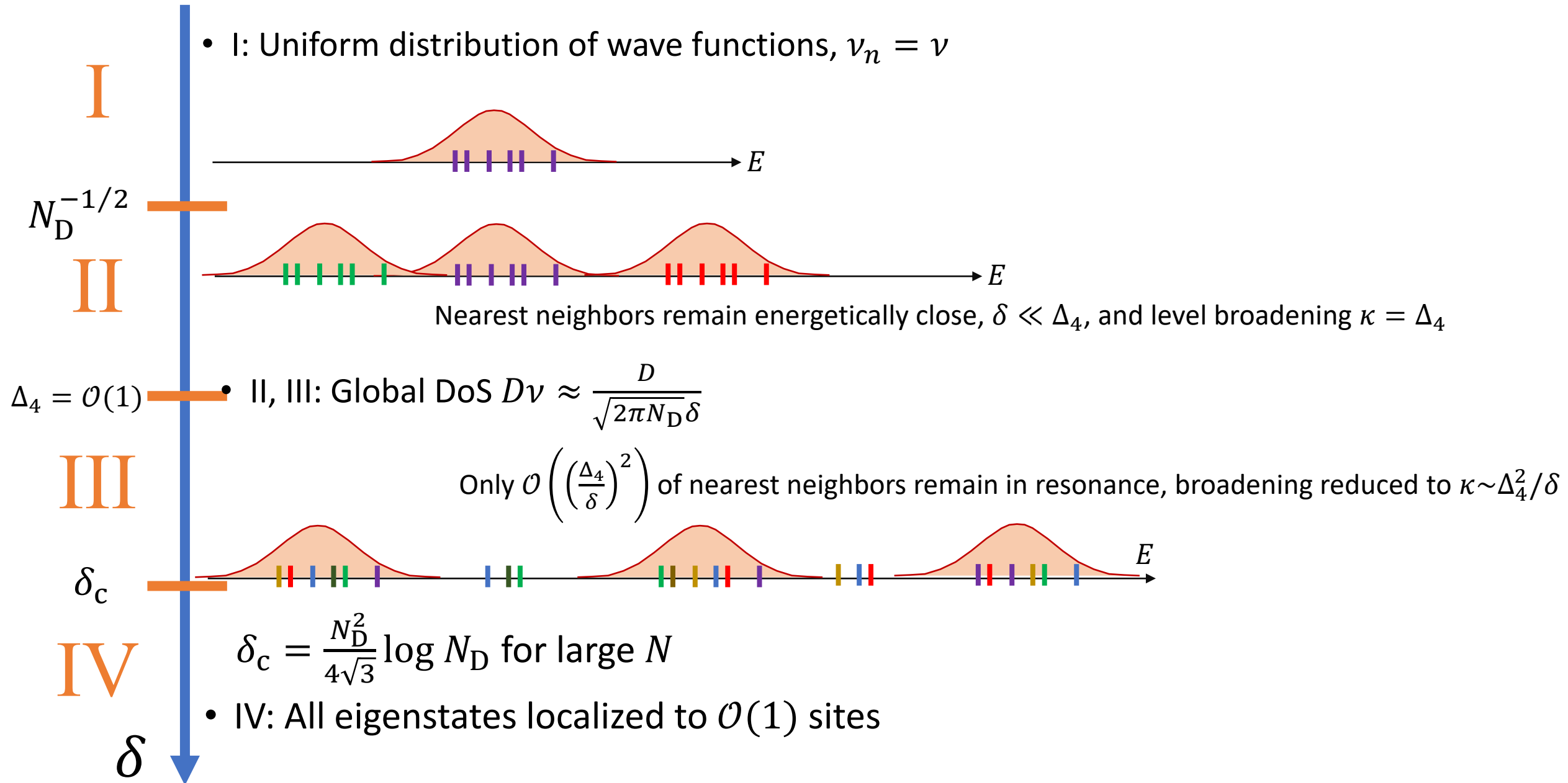


$$n^1 = \bar{n}^1, n^2 = \bar{n}^2, n^3 = \bar{n}^3, n^4 = \bar{n}^4, n^5 = \bar{n}^5$$

$$n^1 = \bar{n}^1, \mathbf{n}^2 = \bar{\mathbf{n}}^4, n^3 = \bar{n}^3, \mathbf{n}^4 = \bar{\mathbf{n}}^2, n^5 = \bar{n}^5$$

$$M_r = \langle \text{tr}_A(\rho_A^r) \rangle = \sum_{\sigma} \sum_{\mathcal{N}} \prod_{i=1}^r \langle |\psi_{n^i}|^2 \rangle \delta_{\mathcal{N}_A, (\sigma \circ \tau) \mathcal{N}_A} \delta_{\mathcal{N}_B, \sigma \mathcal{N}_B}$$

Analytical results



Regime I: maximally random case

$$D_{A(B)} = 2^{N_{A(B)}-1}$$

Uniform distribution of wave functions, $v_n = v$

$$M_r = \langle \text{tr}_A(\rho_A^r) \rangle, S_A = -\partial_r M_r|_{r=1}$$

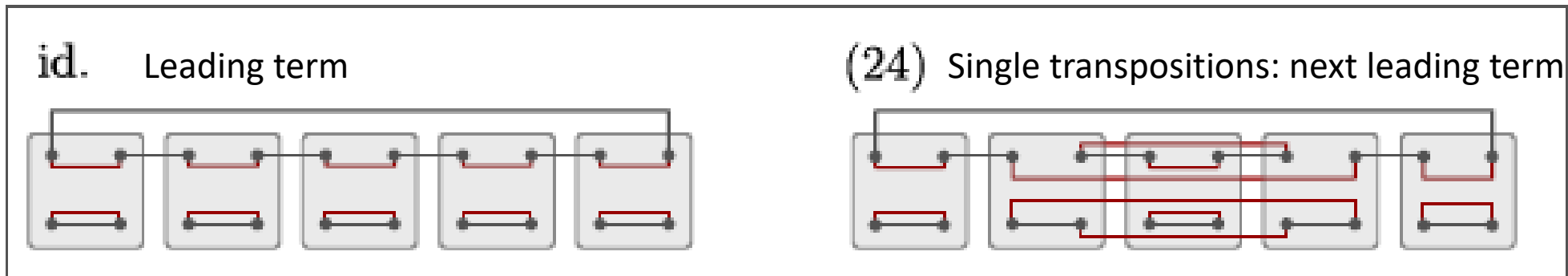
$$M_r \approx D_A^{1-r} + \binom{r}{2} D_A^{2-r} D_B^{-1}$$

Up to single transpositions

Difference from the thermal value $S_{\text{th}} = \ln D_A$

$$S_A - S_{\text{th}} = -\frac{D_A}{2D_B}$$

Exponentially small if $N_A \ll N_B$;
 S_A very close to the thermal value



uniform

$$M_r = \langle \text{tr}_A(\rho_A^r) \rangle = \sum_{\sigma} \sum_{\mathcal{N}} \prod_{i=1}^r \langle |\psi_{n^i}|^2 \rangle \delta_{\mathcal{N}_A, (\sigma \circ \tau) \mathcal{N}_A} \delta_{\mathcal{N}_B, \sigma \mathcal{N}_B}$$

Regimes II and III: reduced effective dimension

- Assume ergodicity and calculate S_A
- Energy shell: extended cluster of resonant sites (width κ) embedded in the Fock space
- Neighboring sites of n : energy $v_m = v_n \pm \mathcal{O}(\delta)$, much more likely to be in the same shell because $\delta \ll \Delta_2 = \sqrt{N_D} \delta$

Additional assumptions

- Exponentially large number of sites $\rightarrow$ self averaging
(sum over site energies = average over approx. Gaussian distributed contributions of subsystem energies to the total energy)
- Total energy $E \sim E_A + E_B$

$\rightarrow$ Up to single transpositions (justified in $1 \ll N_A \ll N_D$ & replica limit):

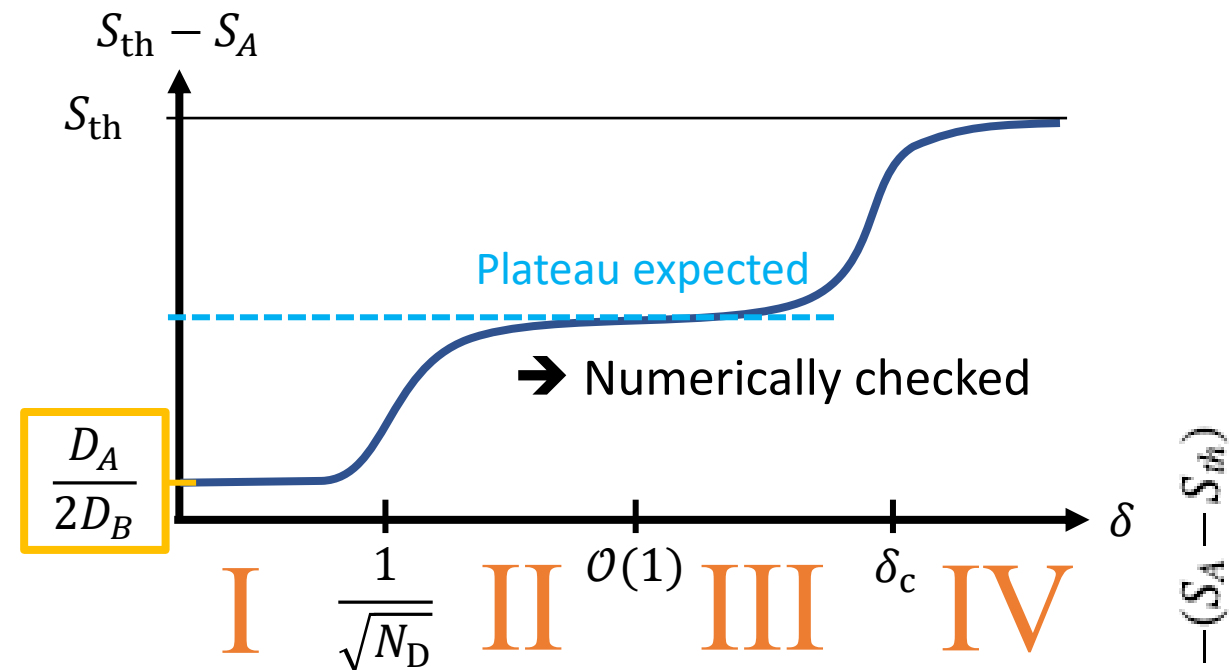
$$S_A - S_{\text{th}} = -\frac{1}{2} \ln \left(\frac{N_D}{N_B} \right) + \frac{N_A}{2N_D} - \sqrt{\frac{N_D}{2N_A} \frac{D_A}{2D_B}} \quad \text{in Regimes II, III}$$

$(\frac{1}{\sqrt{N_D}} \ll \delta < \delta_c \sim N_D^2 \ln N_D)$

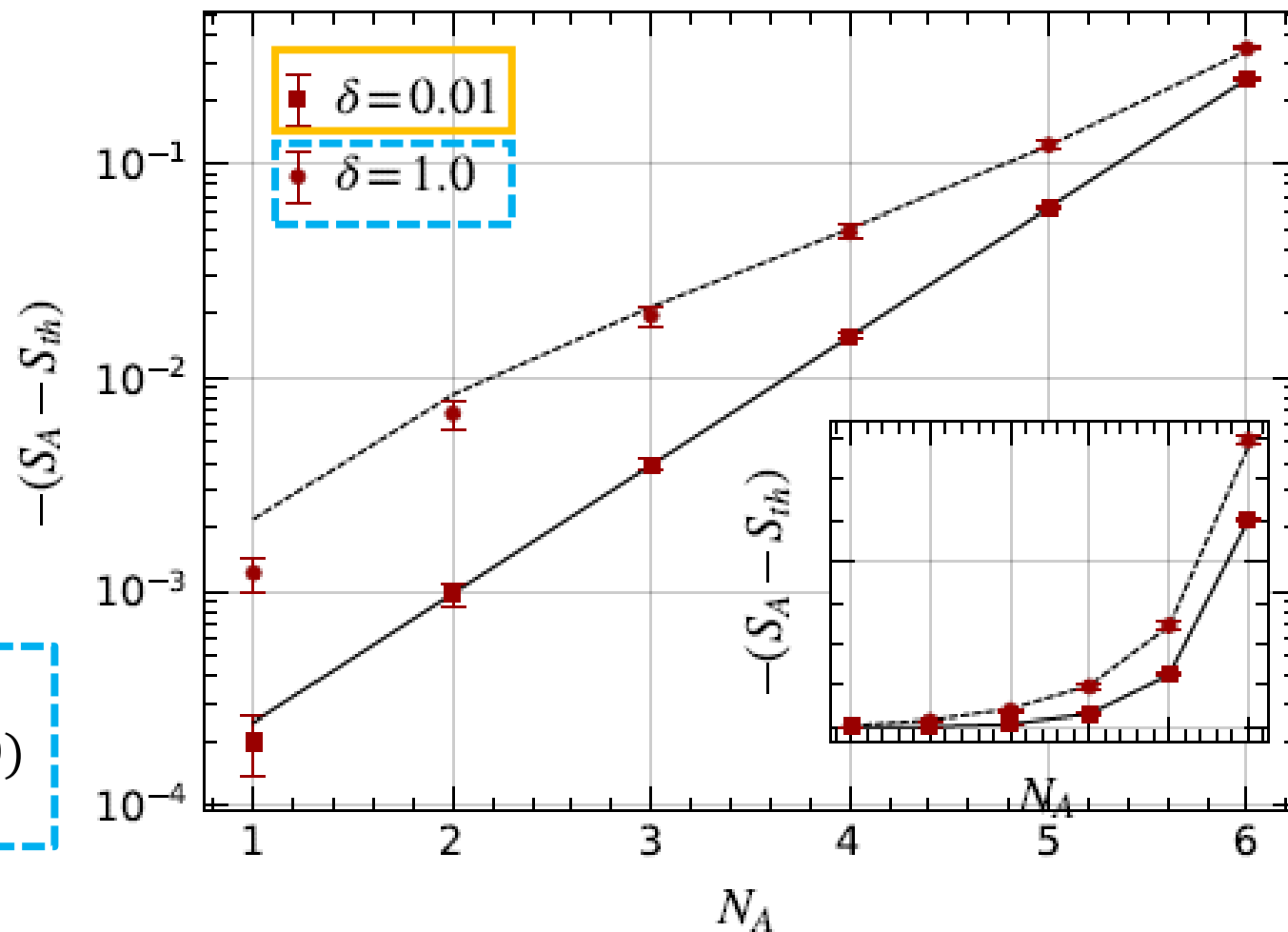
$$S_A - S_{\text{th}} = -\frac{D_A}{2D_B}$$

in Regime I

Offset from the thermal value



$N_D = 14$ ($N = 28$ Majorana fermions)



$$S_A - S_{\text{th}} = -\frac{1}{2} \ln \left(\frac{N_D}{N_B} \right) + \frac{N_A}{2N_D} - \sqrt{\frac{N_D}{2N_A} \frac{D_A}{2D_B}} \quad (< 0)$$

in Regimes II, III ($\frac{1}{\sqrt{N_D}} \ll \delta < \delta_c \sim N_D^2 \ln N_D$)

Summary

- The Sachdev-Ye-Kitaev (SYK) model: quantum mechanical model realizing chaos bound ($\sim$ random matrix, black holes)
- Several experimental proposals, small systems realized
- SYK₄₊₂: analytically tractable model for many-body localization (MBL)
 - Fock space: $(N/2)$ -dimensional hypercube
- Analytical results on eigenfunction moments and MBL point
 - ➔ Agreement with numerical results without free parameters
- Evaluation of entanglement entropy S_A assuming ergodicity in energy shells
 - ➔ Agreement between the numerical and analytical results