Quantum Tunneling in Nonintegrable Systems and Complex Dynamical Systems

非可積分系のトンネル効果と複素力学系

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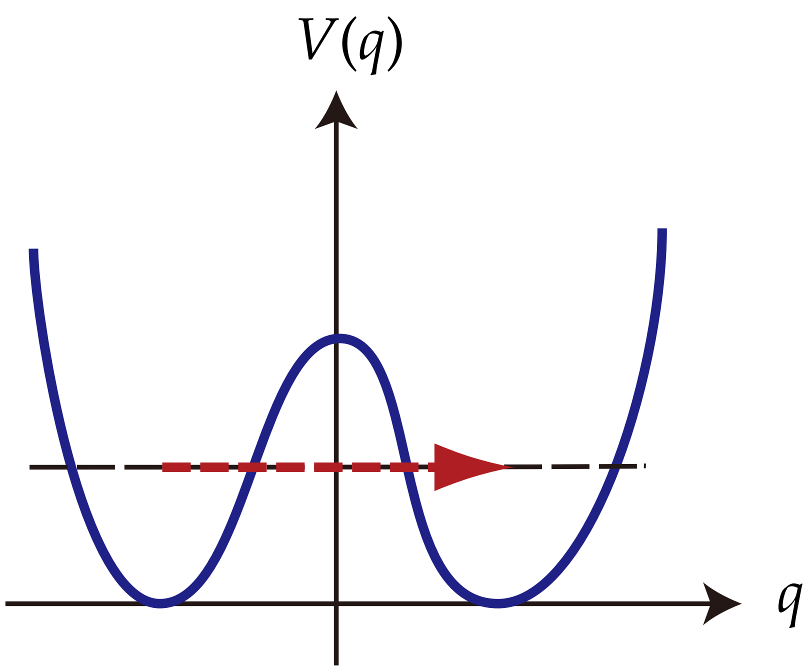
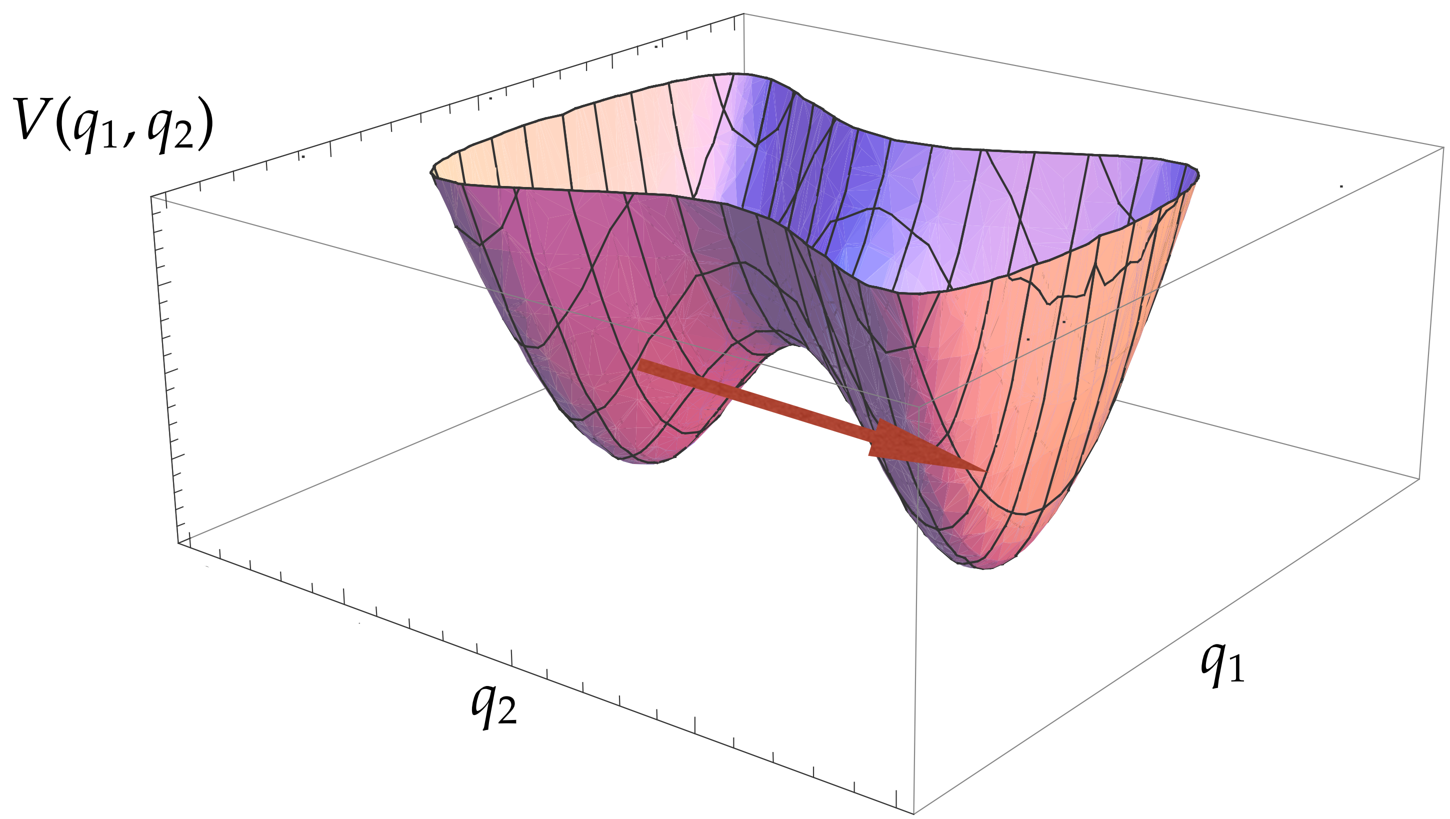
首藤 啓（都立大）

Quantum tunneling

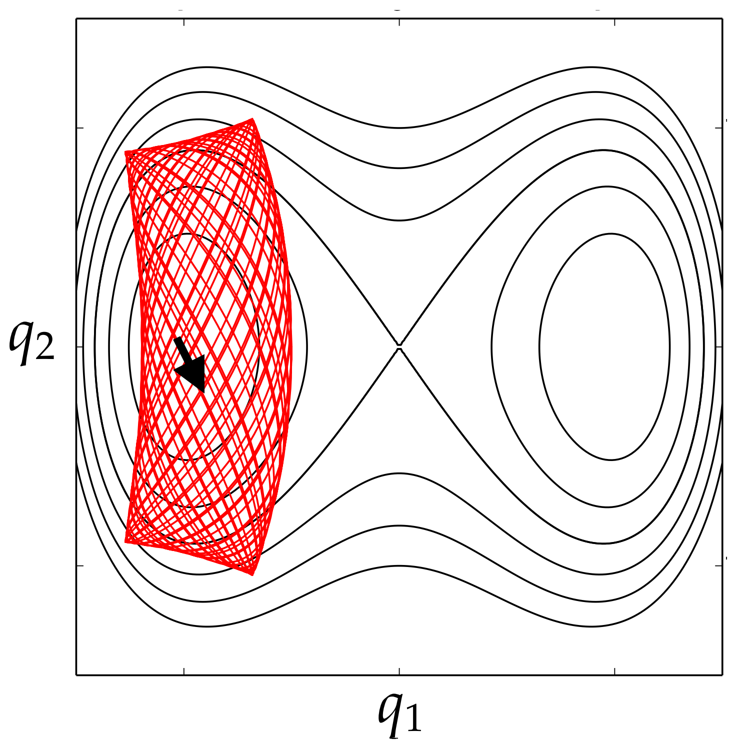
One-dimension



Multi-dimension



Classical dynamics in nonintegrable systems



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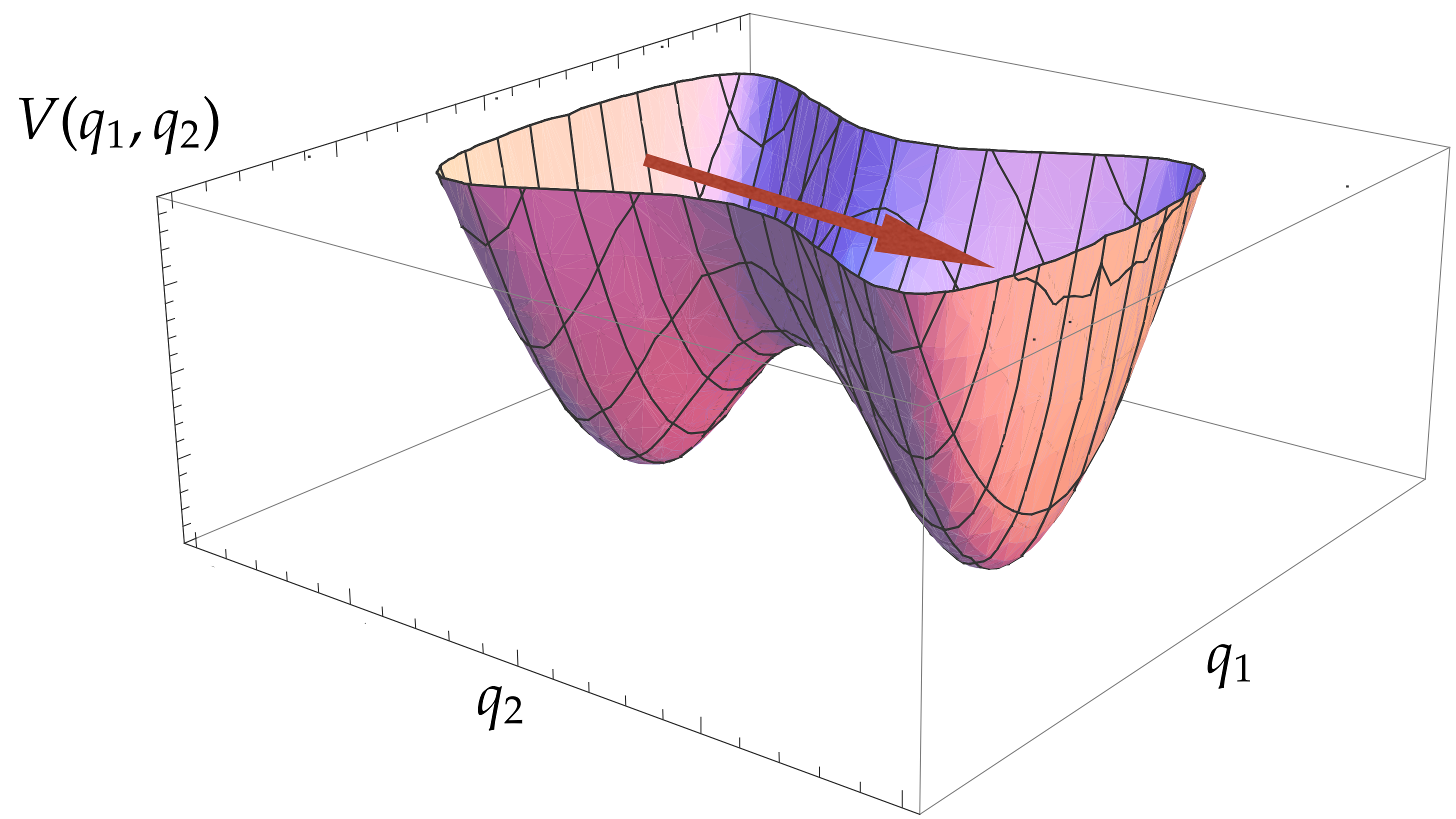
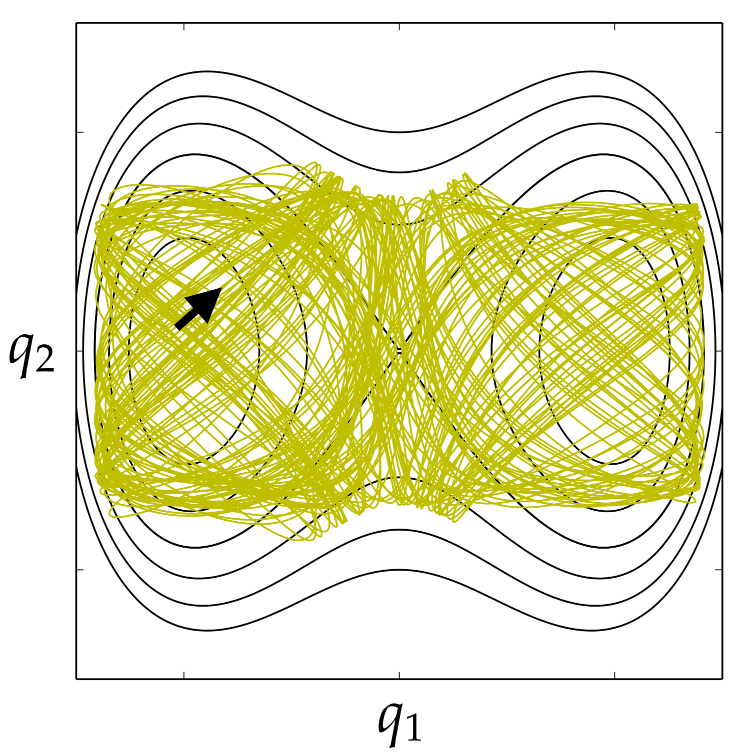
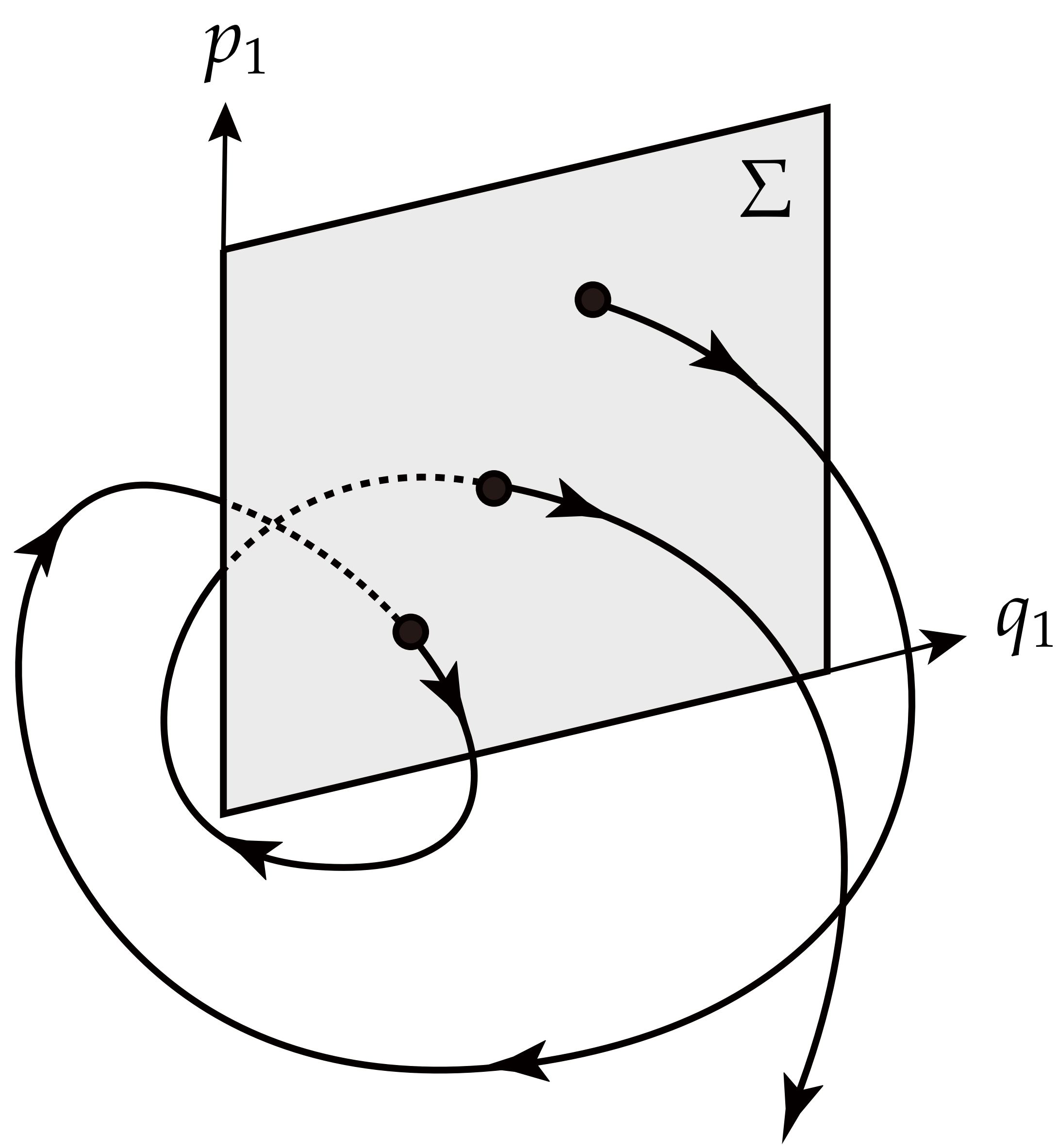




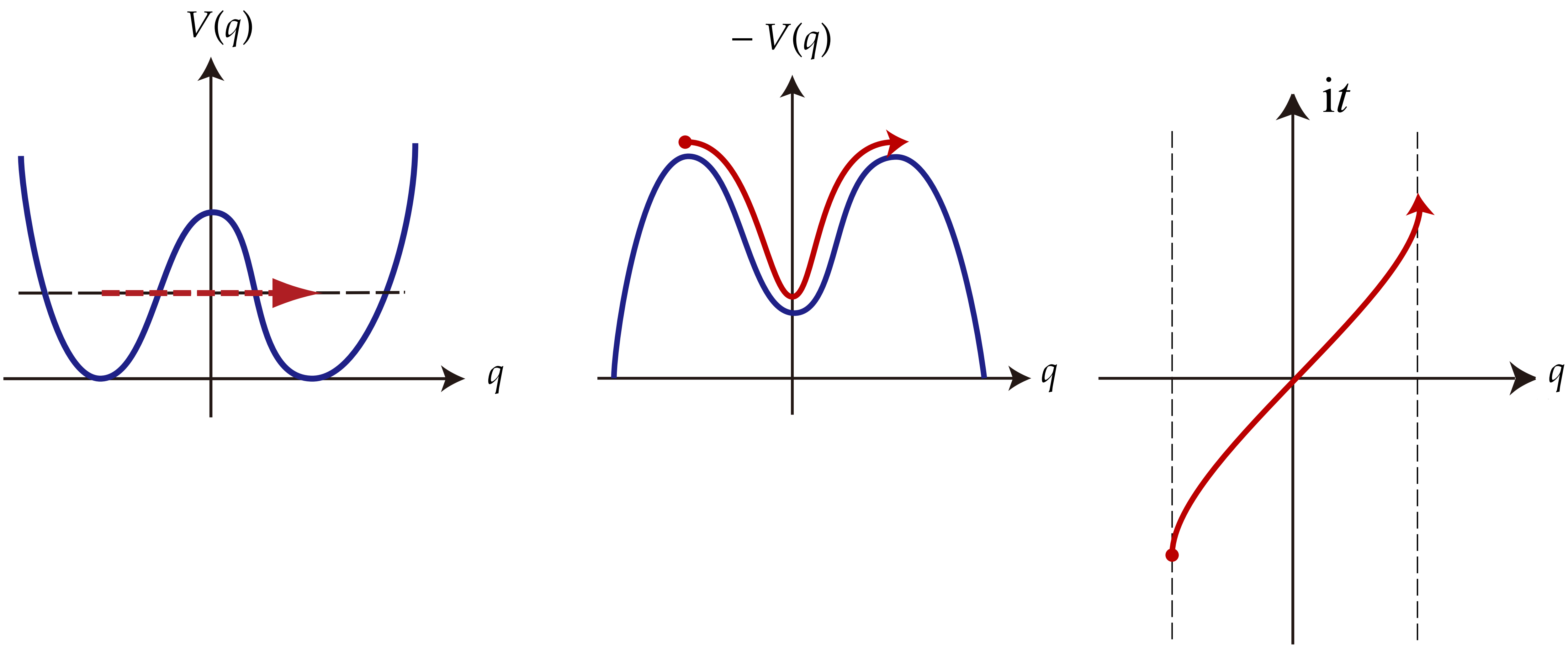
Poincaré section



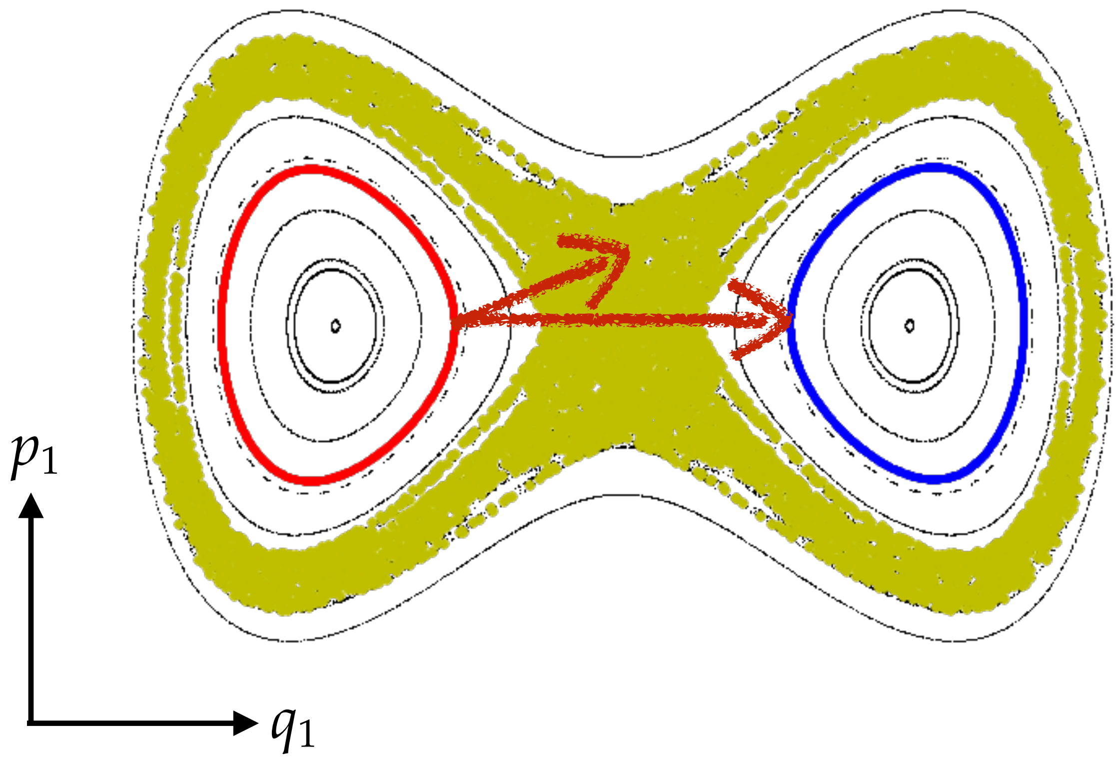
Trajectory on a constant energy surface $H(q_1, q_2, p_1, p_2) = E$

Quantum tunneling and complex path

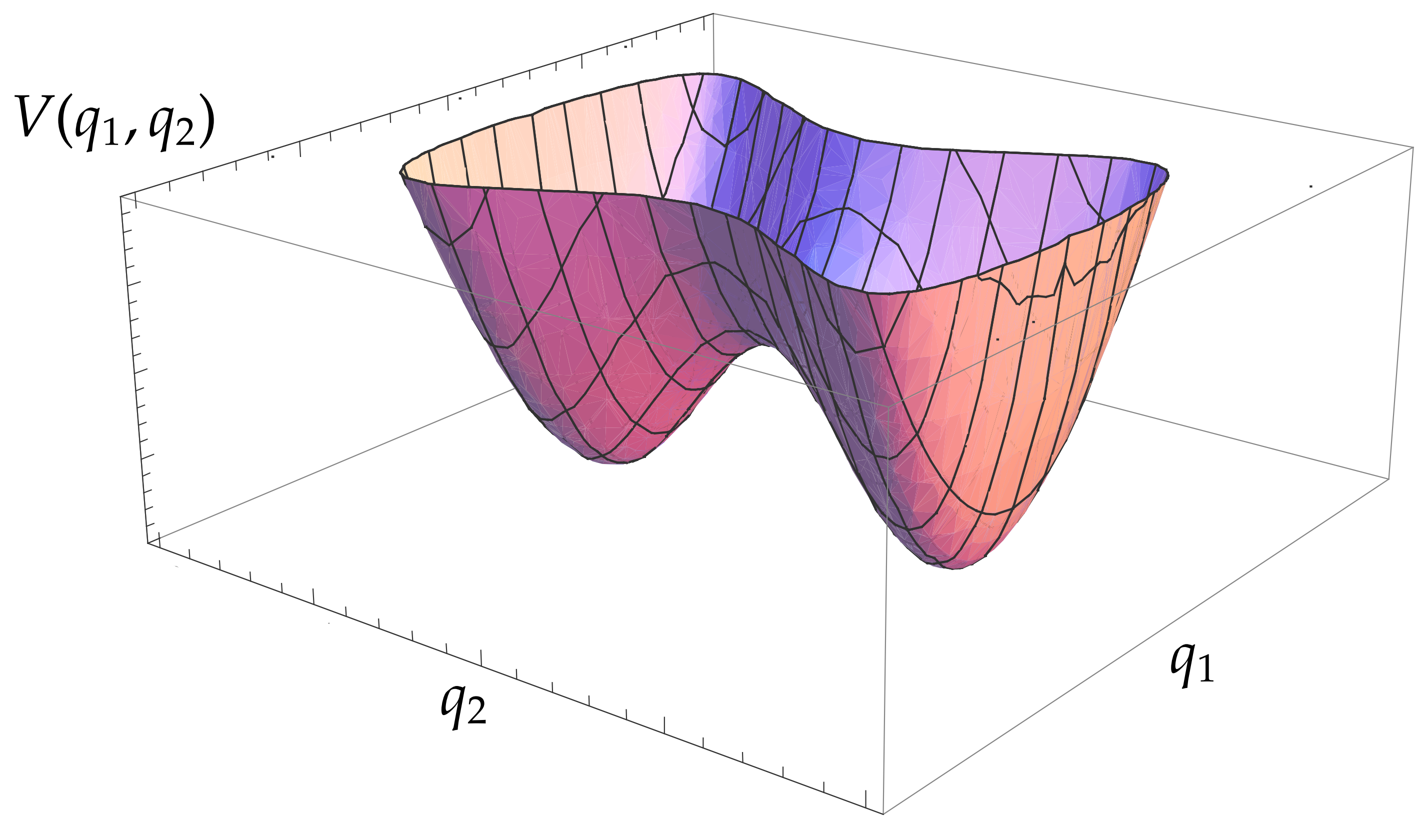
Instanton



Dynamical tunneling and complex paths

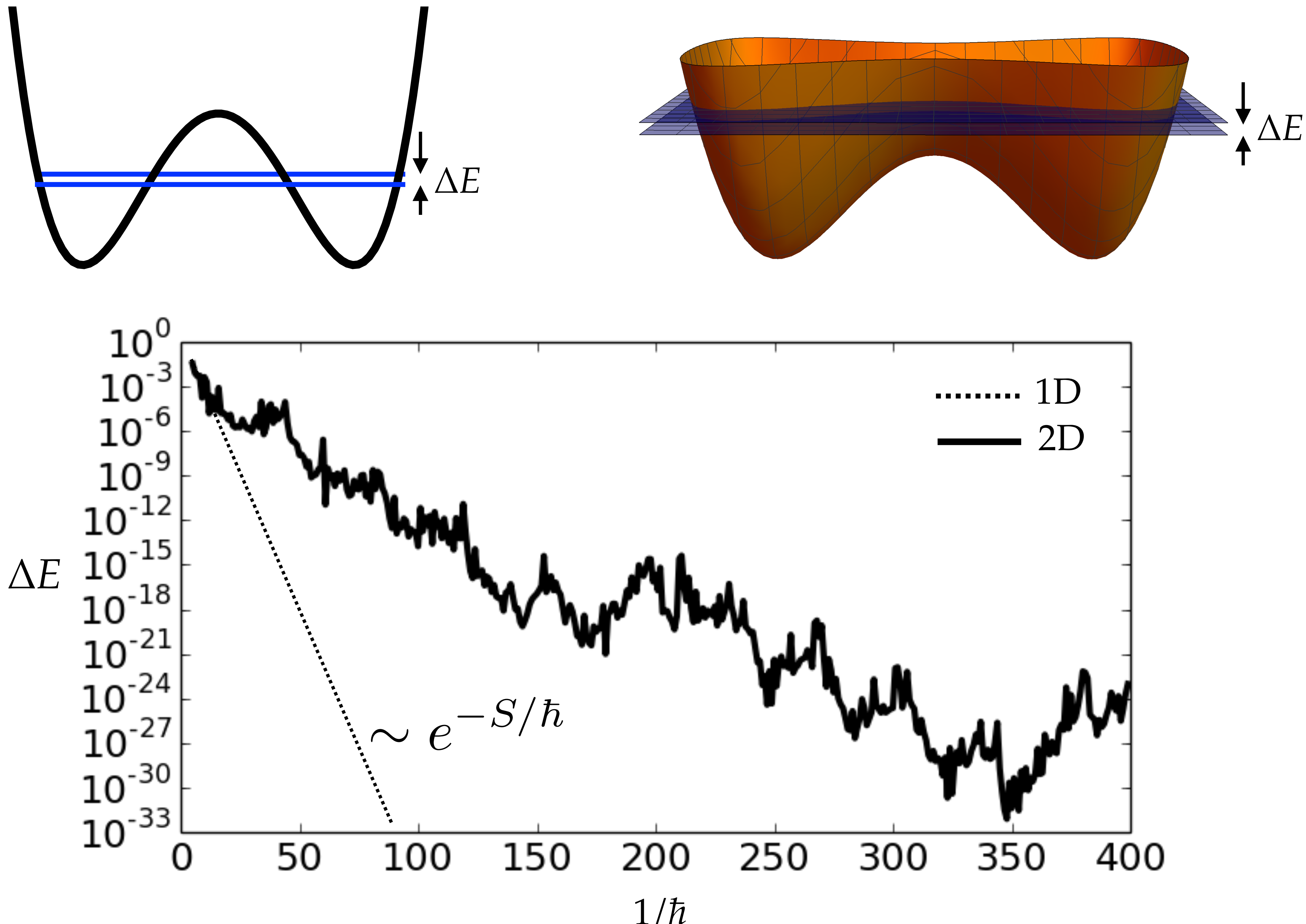


Poincaré section



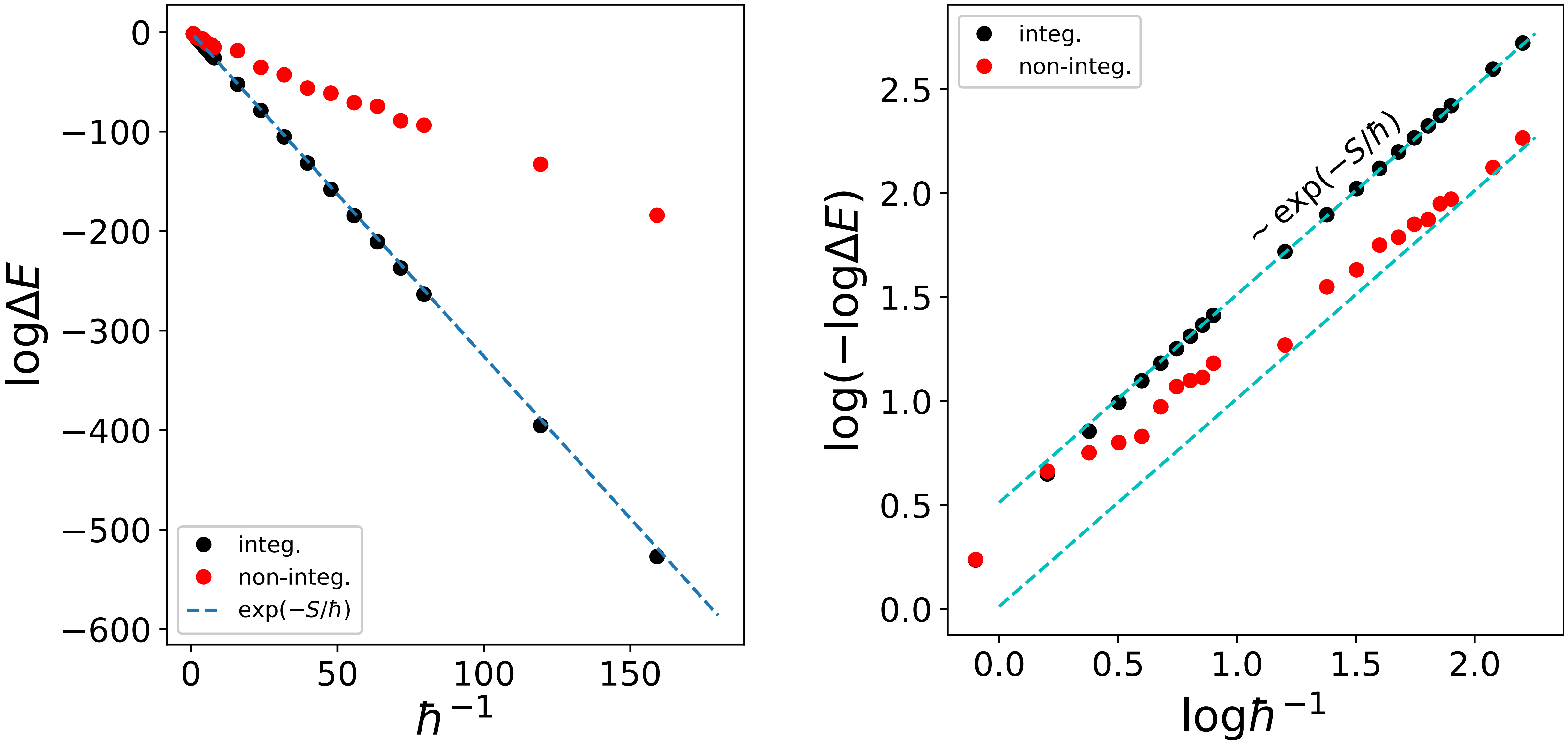
How are disconnected regions connected ?

Tunneling splitting in 1D and 2D



Y. Hanada et al (2021)

Stretched exponential decay of tunneling splittings



Y. Hanada et al (2021)

Poincaré section and area-preserving maps

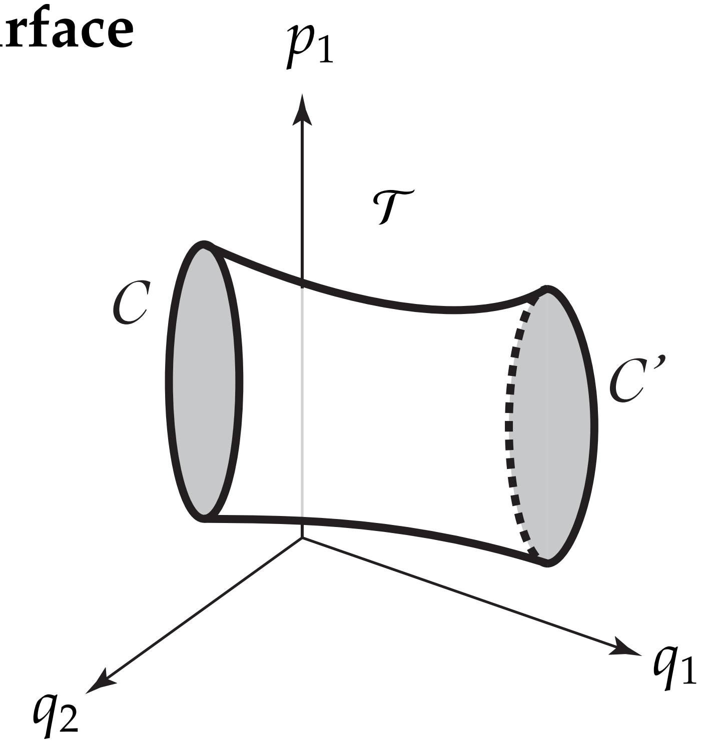
2-D autonomous Hamiltonian $H(q_1, q_2, p_1, p_2)$

Area enclosed by a closed loop C on the constant energy surface

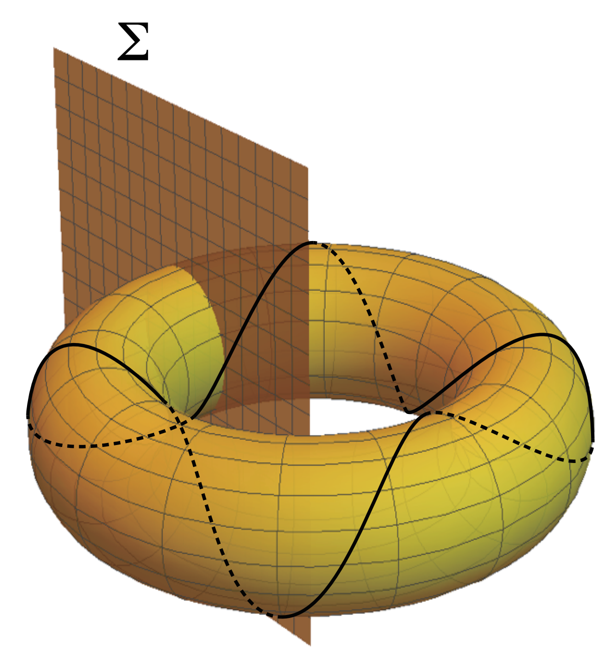
$$S[C] = \oint_C p \cdot dq$$

is preserved along the Hamiltonian flow:

$$S[C] = S[C']$$



Poincaré map $F : \Sigma \mapsto \Sigma$ is area-preserving (symplectic)



Area-preserving map and mixed phase space

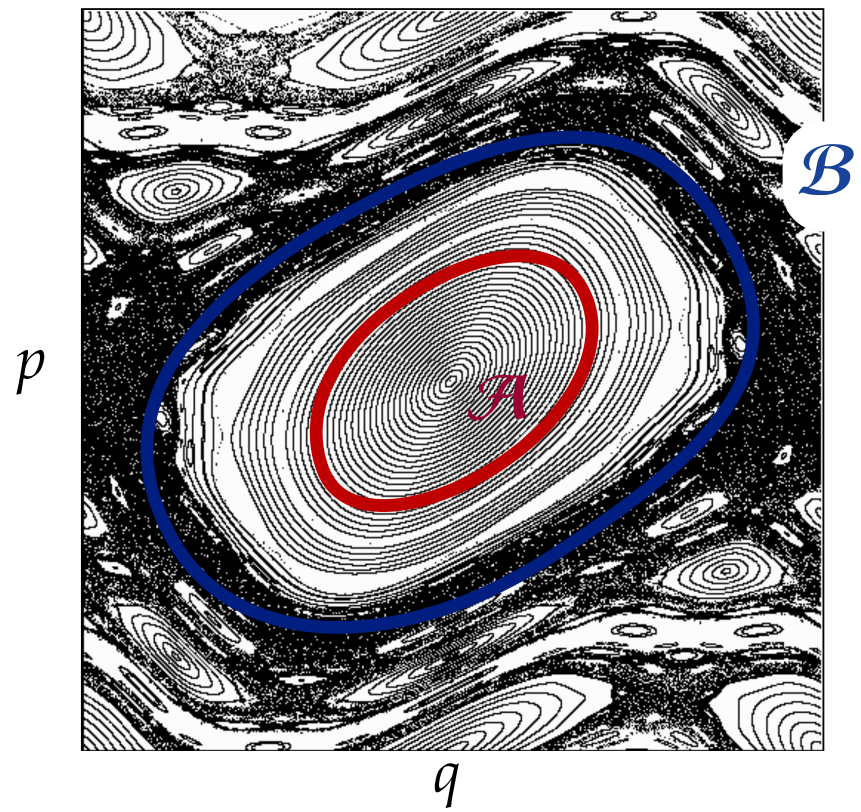
Area-preserving map

$$F : \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p - V'(q) \\ q + p' \end{pmatrix}$$

Kicked rotor

$$H(q,p,t) = \frac{1}{2}p^2 + V(q) \sum_{n=-\infty}^{\infty} \delta(t-n)$$

$V(q) = K \sin q$



Area-preserving map and mixed phase space

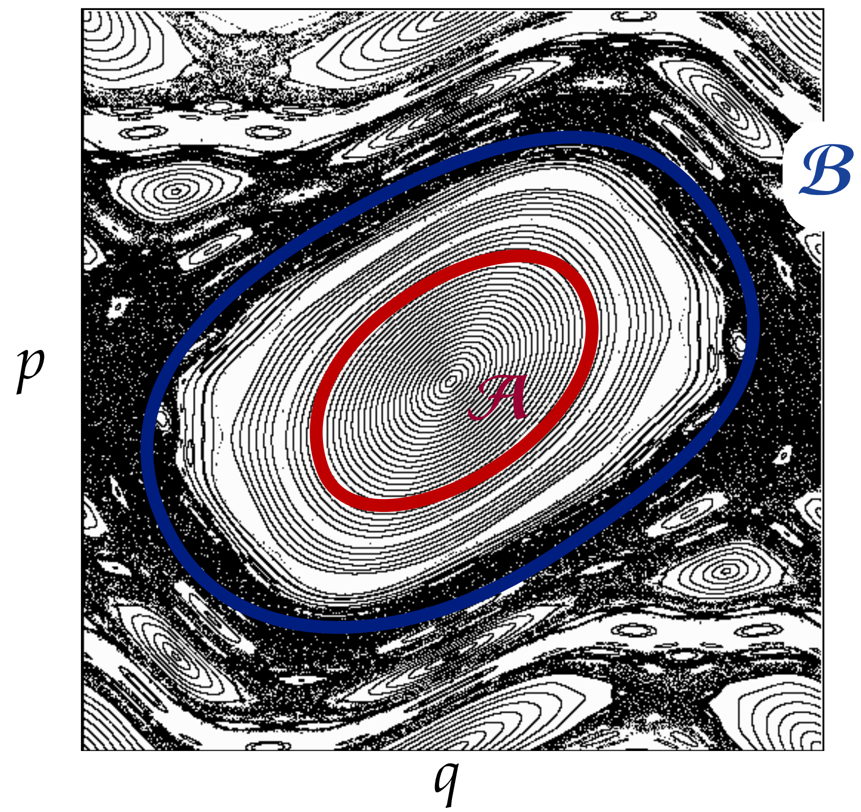
Area-preserving map

$$F : \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p - V'(q) \\ q + p' \end{pmatrix}$$

Forbidden process in classical dynamics

$\mathcal{A} \cap F^{-n}(\mathcal{B}) = \emptyset$ **for** $\forall n$, **if** $\mathcal{A}, \mathcal{B} (\in \mathbb{R})$ **are dynamically separated.**

$V(q) = K \sin q$



Area-preserving map and mixed phase space

Quantum map

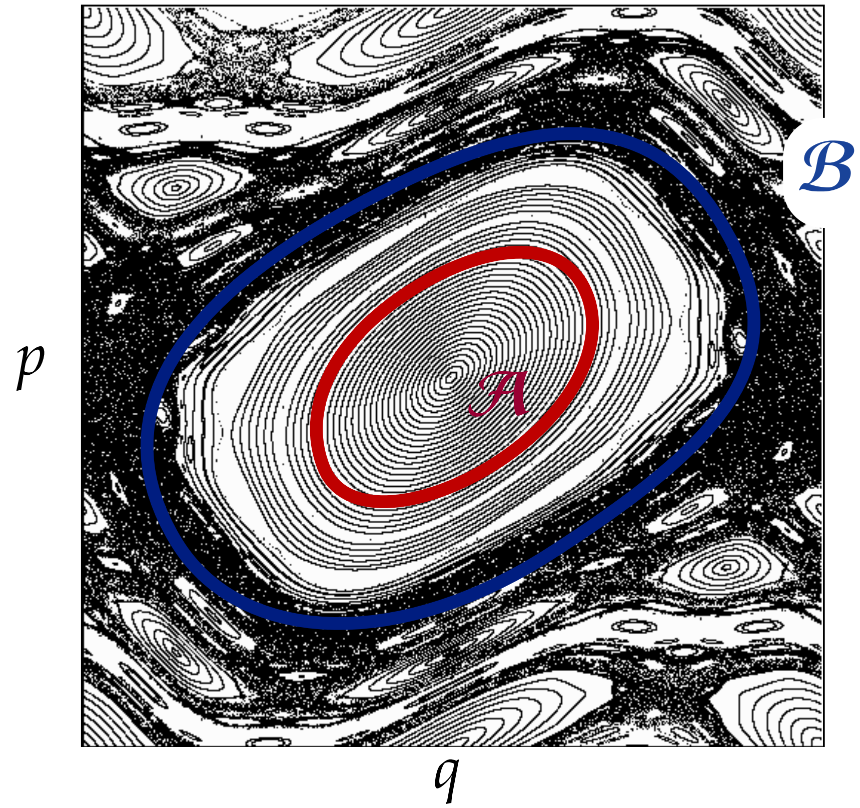
$$\hat{U} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}T(\hat{p})}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V(\hat{q})}$$

Propagator

$$K(a,b) = \langle b|\hat{U}^n|a\rangle = \int\cdots\int \prod_j dq_j \prod_j dp_j \exp\left[\frac{i}{\hbar}S(\{q_j\},\{p_j\})\right]$$

Tunneling process in quantum dynamics

$K(a,b) \neq 0$ even if $\mathcal{A}, \mathcal{B} (\in \mathbb{R})$ are dynamically separated.



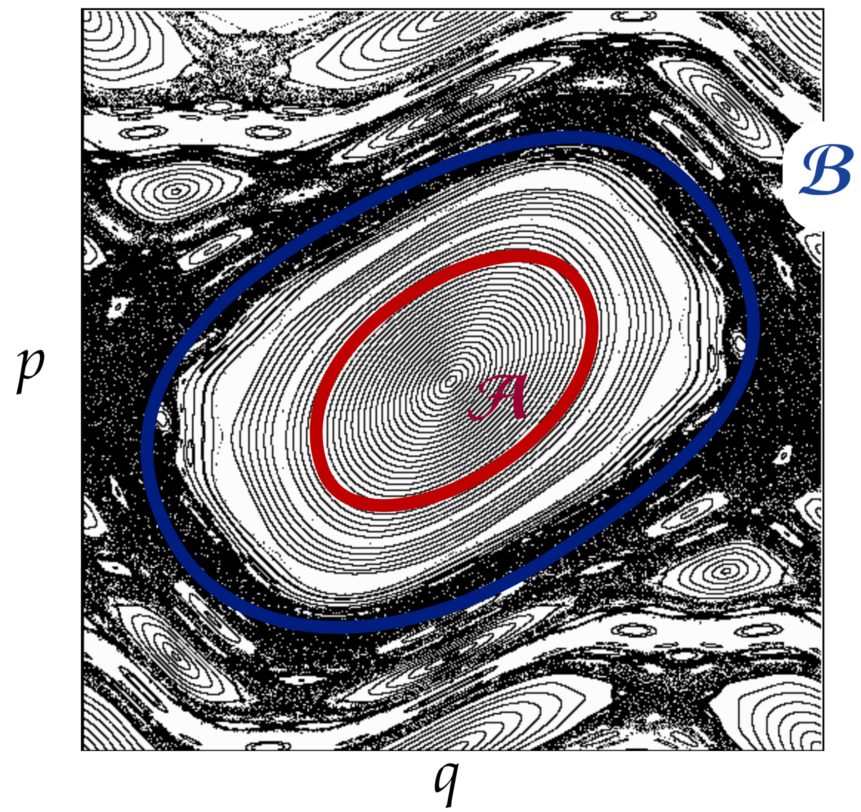
Area-preserving map and mixed phase space

Semiclassical approximation (Van-Vleck, Gutzwiller)

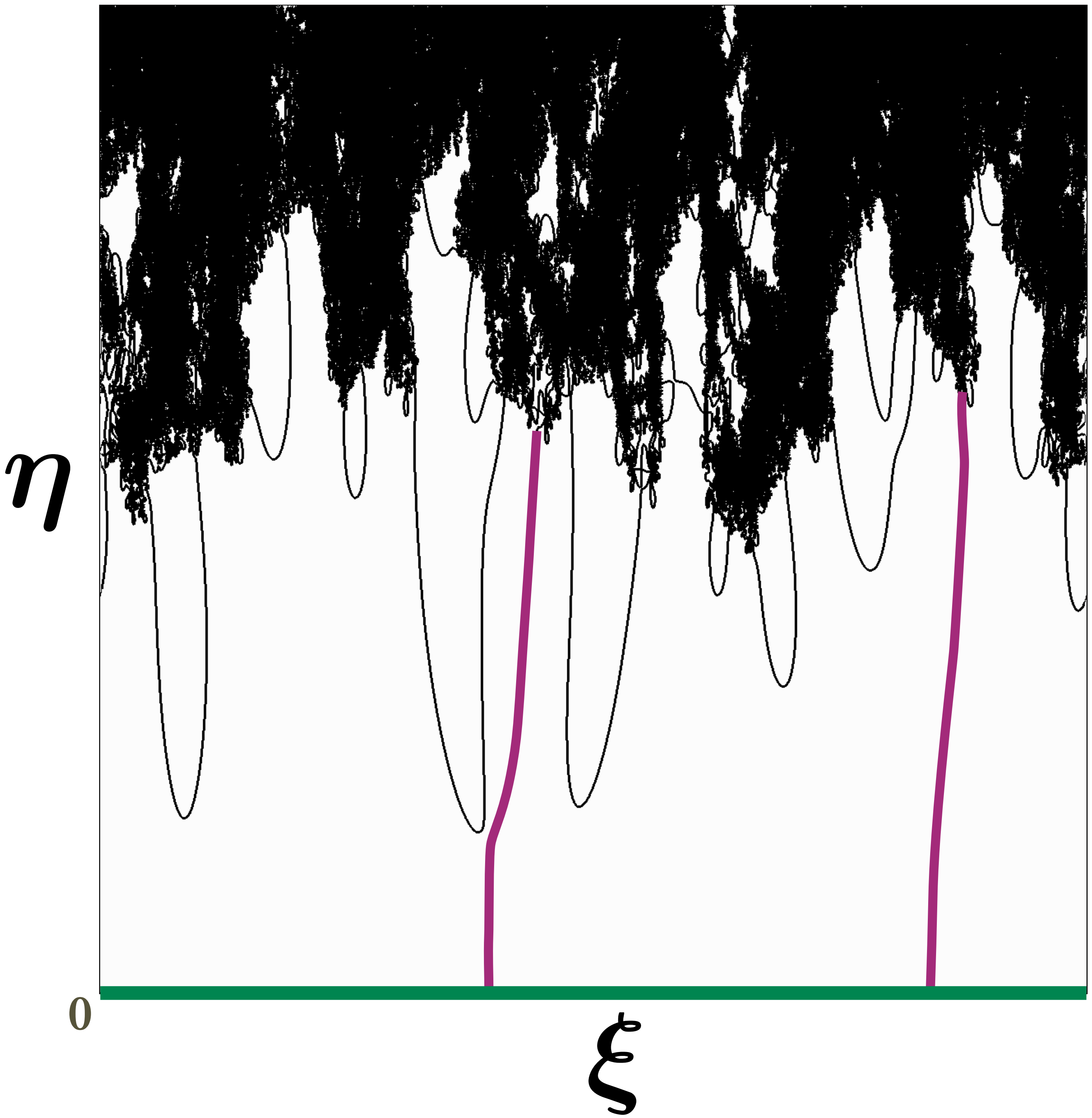
$$K^{sc}(a,b) = \sum_{\gamma} A_n^{(\gamma)}(a,b) \exp\left\{\frac{i}{\hbar} S_n^{(\gamma)}(a,b)\right\}$$

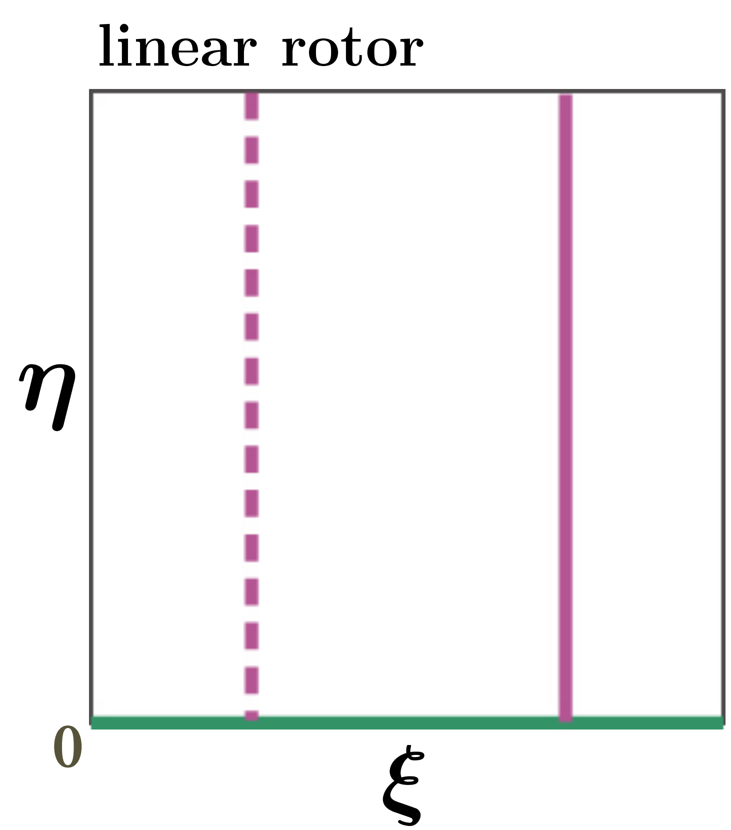
γ: classical orbits connecting a and b

If $F^n(\mathcal{A}) \cap \mathcal{B} = \emptyset$, then γ should be complex.

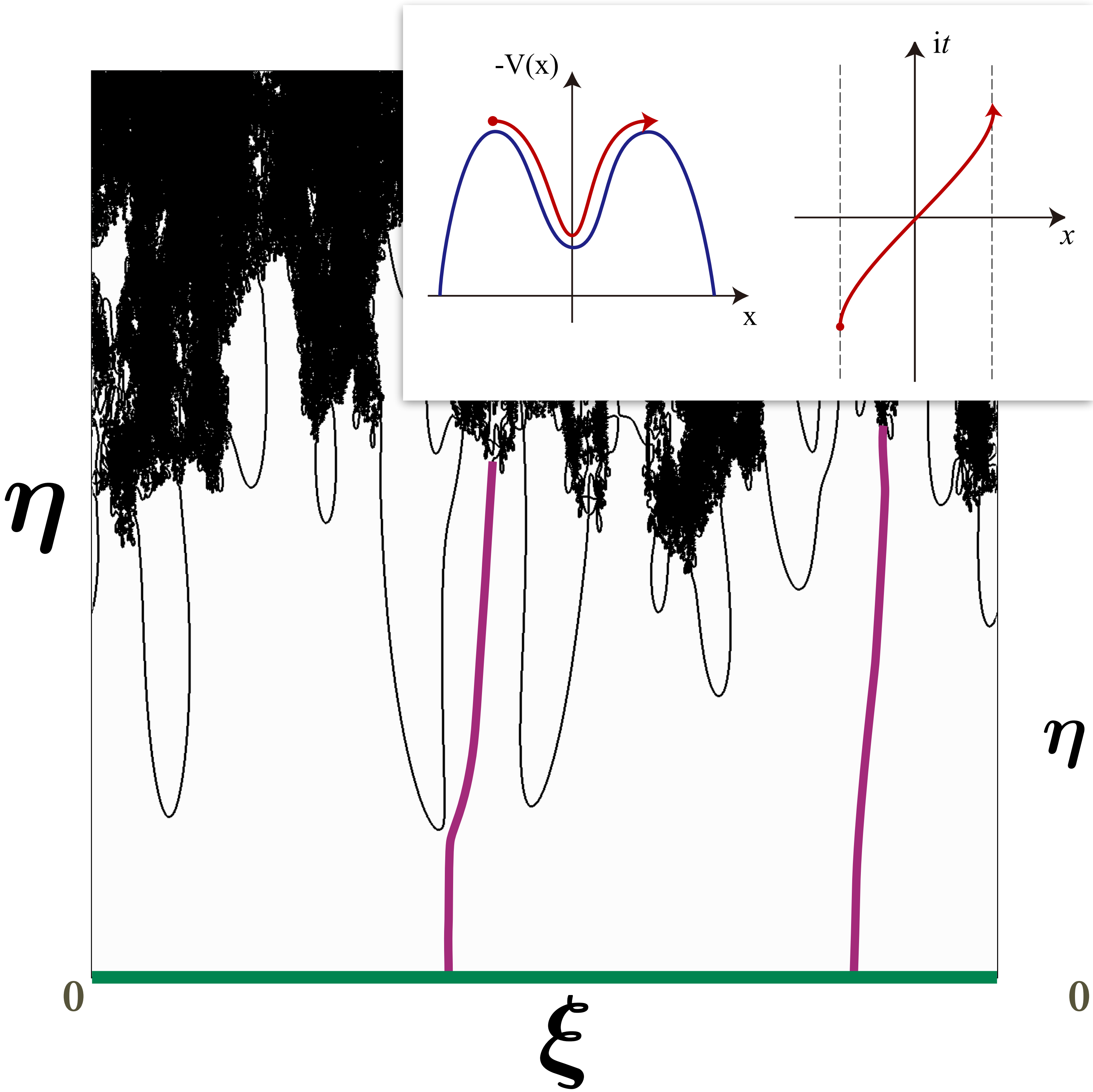


Complex paths contributing the semiclassical propagator

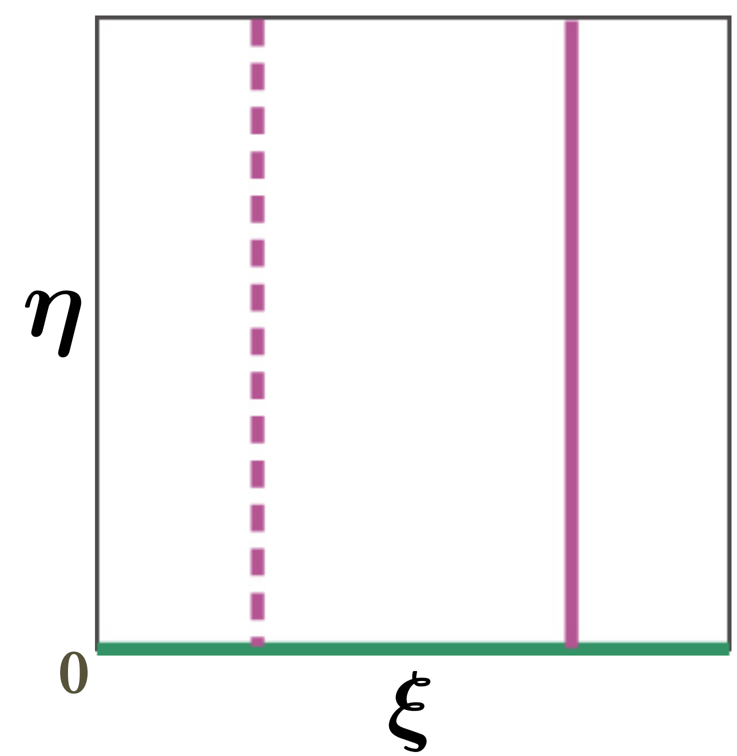




Complex paths contributing the semiclassical propagator



linear rotor



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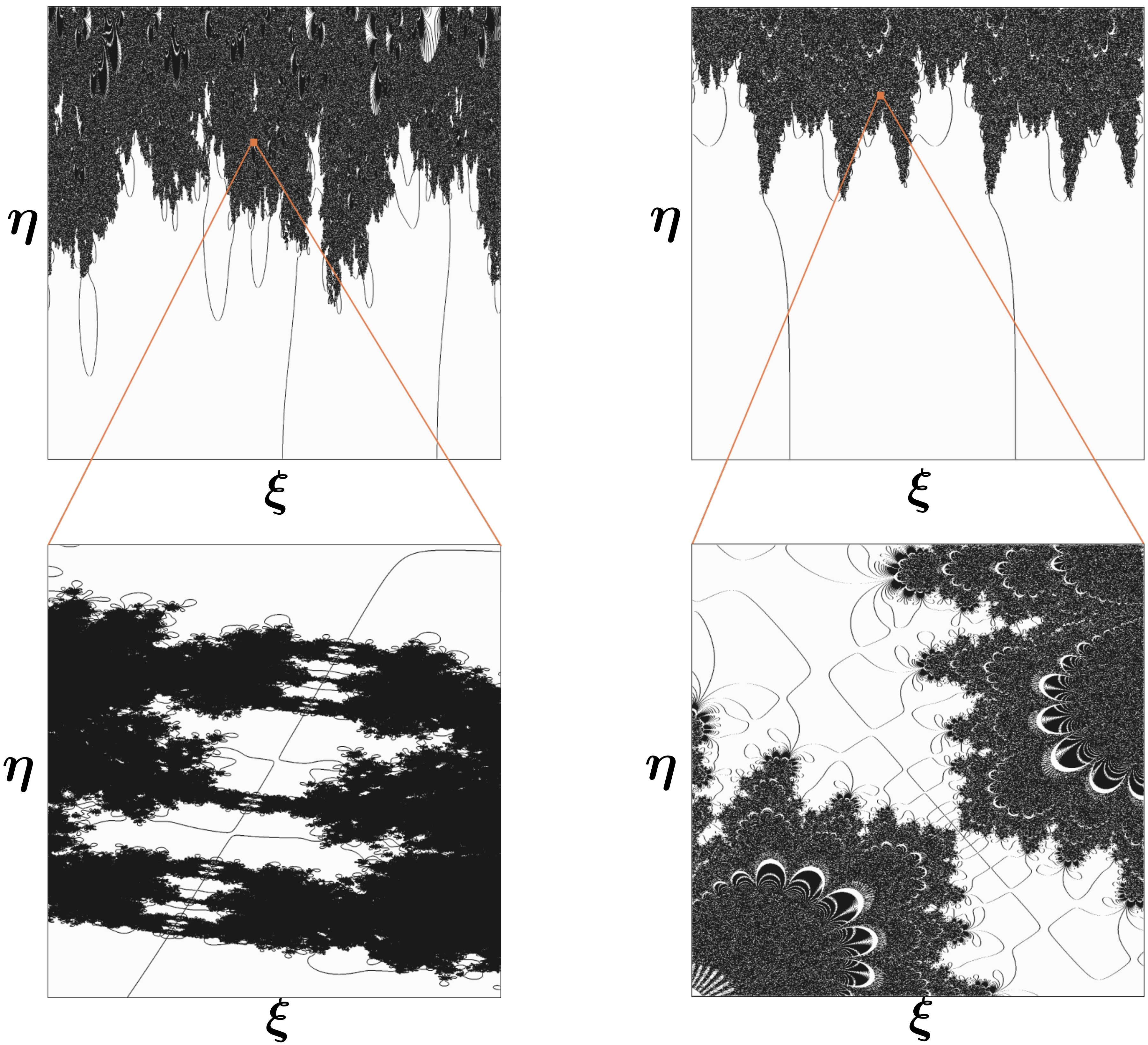
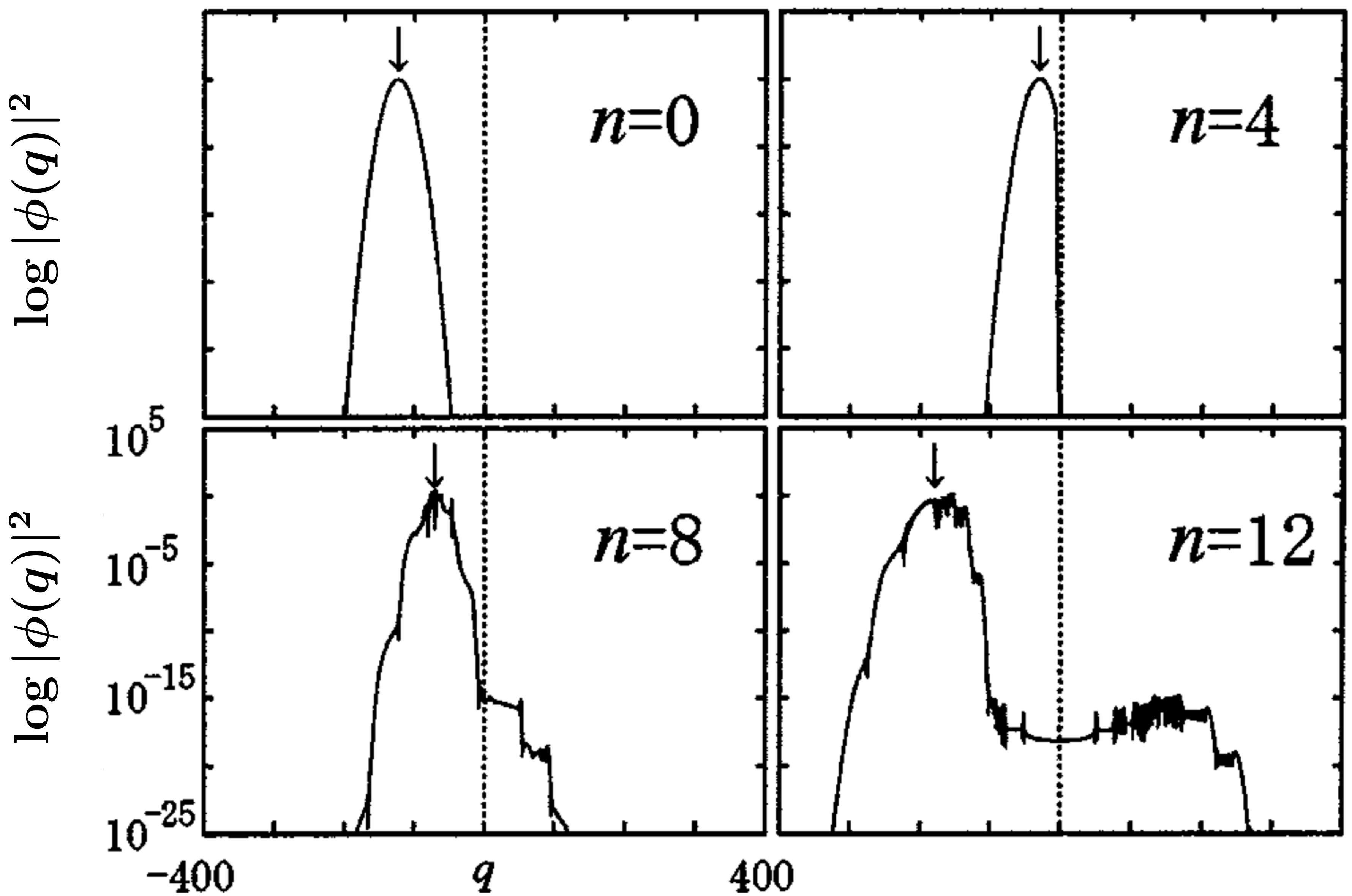


Comparison between quantum and semiclassical (numerics)

Area-preserving scattering map

$$F : \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p - V'(q) \\ q + p' \end{pmatrix} \qquad V(q) = K \exp\left(-\gamma q^2\right)$$



T. Onishi, AS, K. Takahashi and K.S. Ikeada (2003)

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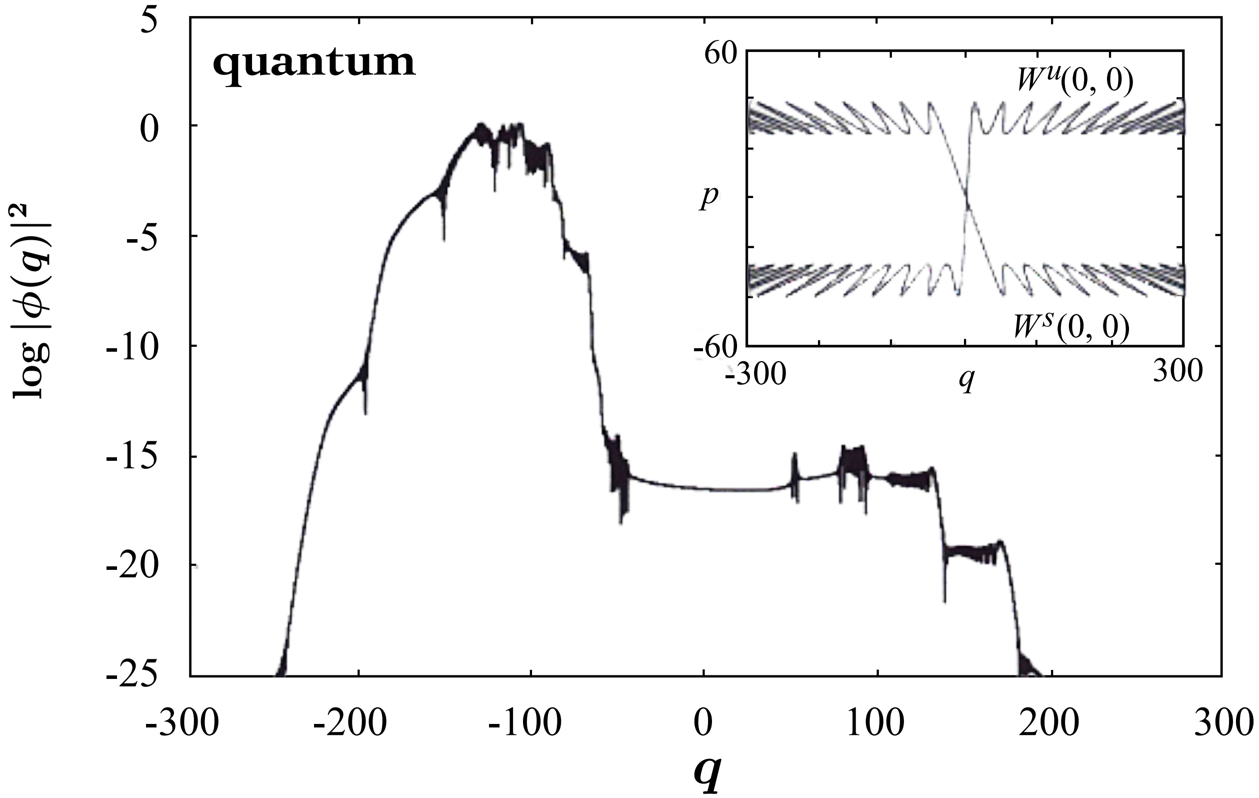

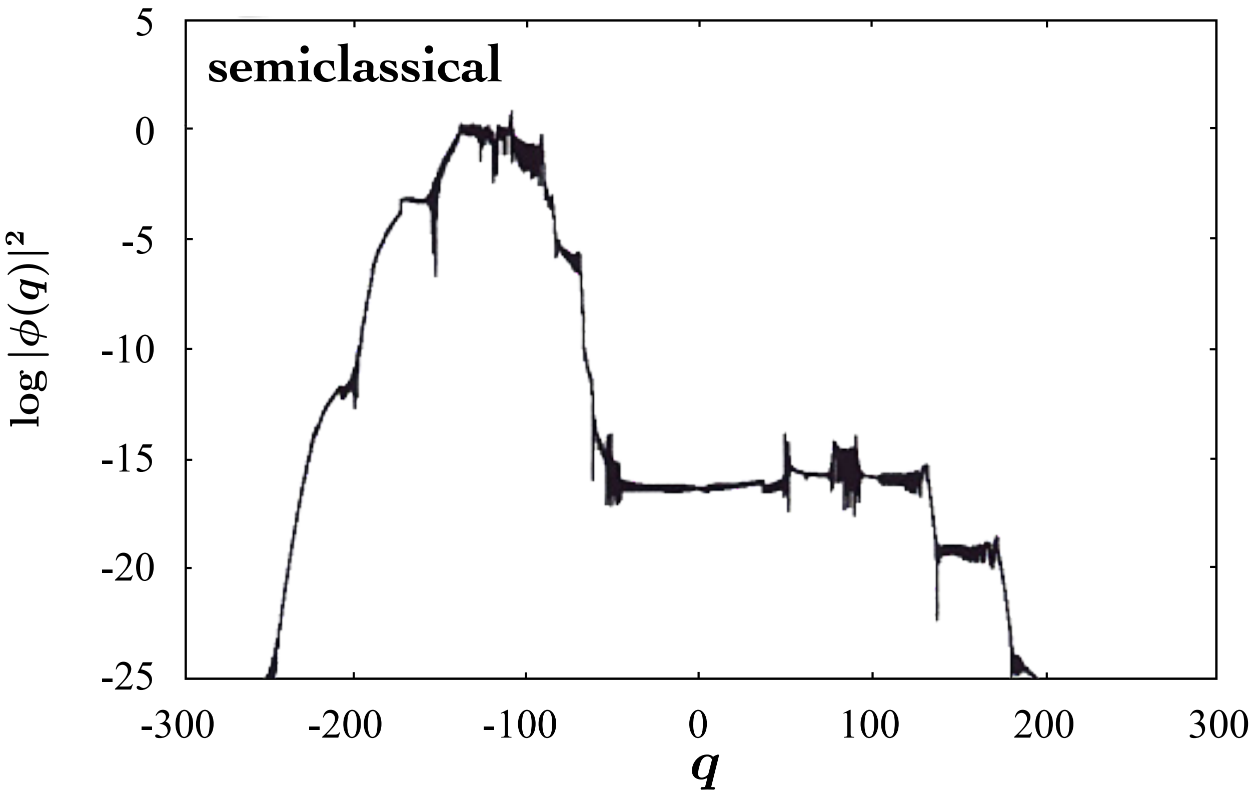

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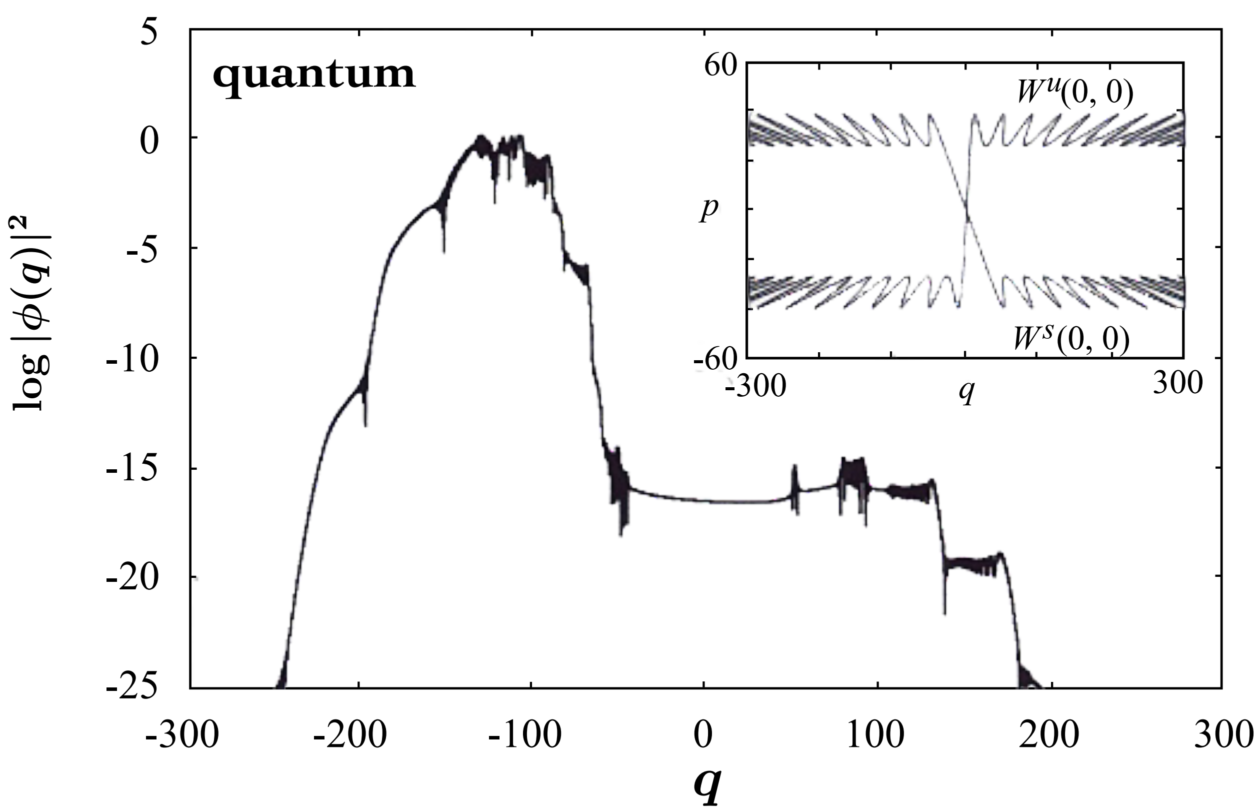

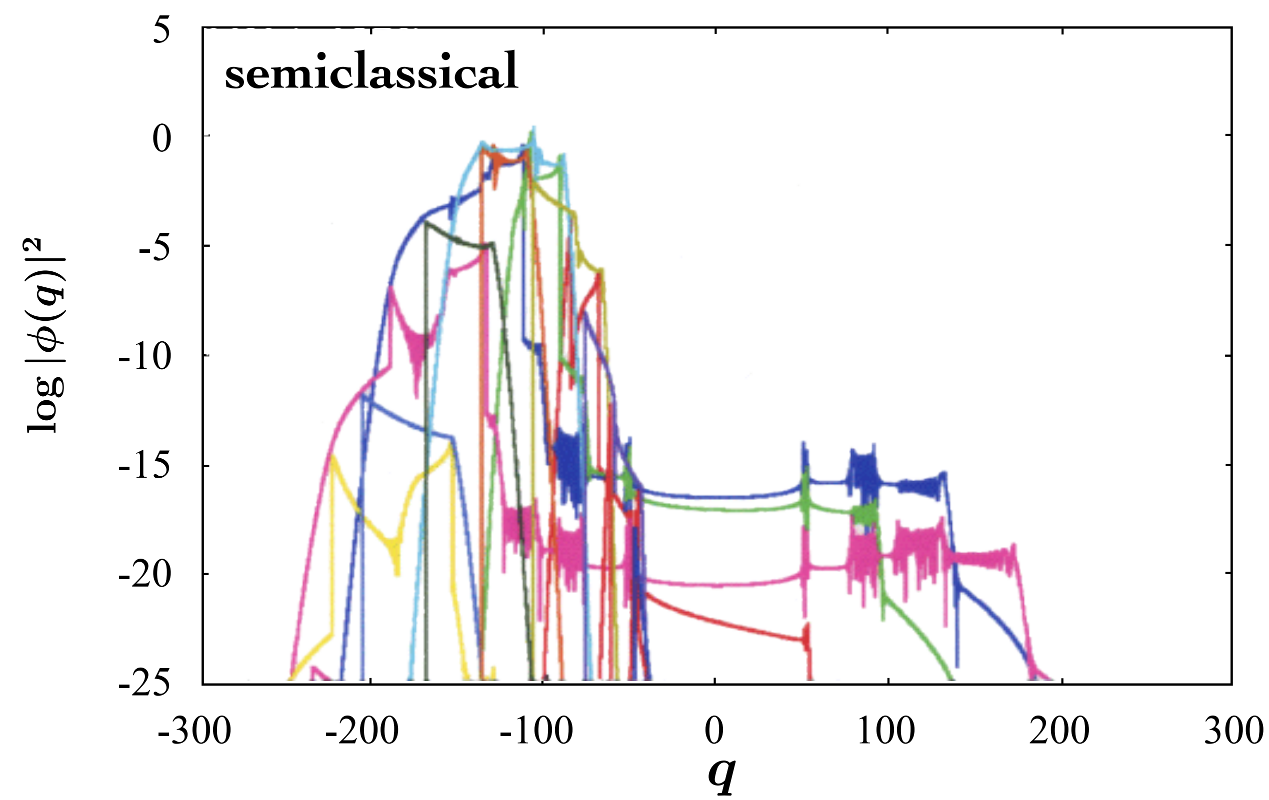

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1-dimensional complex dynamics and the Julia set

1-dimensional polynomial map $F : \mathbb{C} \mapsto \mathbb{C}$

$$F : z \mapsto F(z)$$

where

$$F(z) = z^d + a_1 z^{d-1} + \cdots + a_d \qquad (d \geq 2)$$

Classify the orbits according to the behavior of $n \to \infty$

$I = \{ z \in \mathbb{C} \mid \lim_{n\to\infty} F^n(z) = \infty \}$	:	**The set of escaping points**
$K = \{ z \in \mathbb{C} \mid \lim_{n\to\infty} F^n(z) \text{ is bounded} \}$	:	**Filled − in Julia set**
$F = \mathbb{C} - K$	:	**Fatou set**

In particular

$J = \partial K$	:	**Julia set**

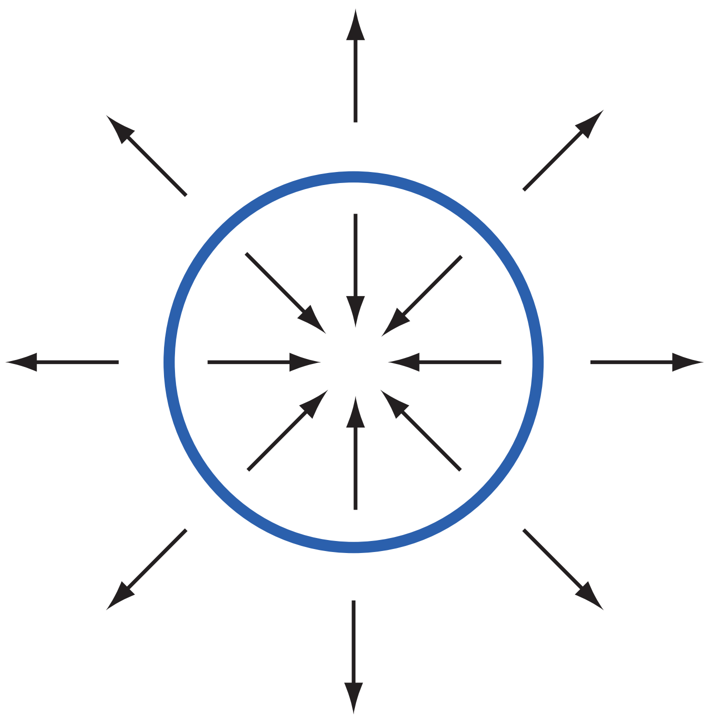
1-dimensional complex dynamics and the Julia set

- $F(z) = z^2$

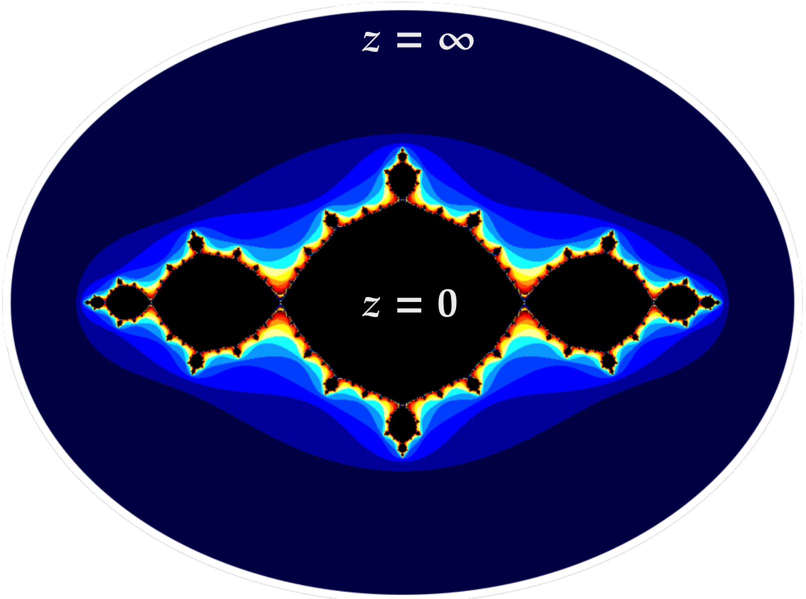
  $I = \{\, |z| > 1 \,\},\quad K = \{\, |z| \leq 1 \,\},\quad J = \{\, |z| = 1 \,\}$

  - $z = 0$ and $z = \infty$ are both attracting fixed points of F.
    The points $z \in I$ tend to ∞ and also the points $z \in K - J$ converge to $z = 0$ monotonically.

  - The orbits $z \in J$ are chaotic.
    Putting $z = e^{2\pi i\theta}$, then the map on J can be reduced to
    $\theta \mapsto 2\theta \pmod 1$.



- $F(z) = z^2 + c$



Dynamics on the Julia set

"P is chaotic on J"

1. **Sensitive dependence on initial conditions**

   there exists $\delta > 0$ such that, for any $z \in J$ and any nbd U of z, there exists $\zeta \in U$ and $n \geq 0$ such that $|F^n(z) - F(\zeta)| > \delta$

2. **Dense periodic repelling periodic orbits**

   $$J = \partial K = \overline{\{ \textit{repelling periodic points} \}}$$

3. **Topological transitivity**

   For any open sets $U, V \subset J$, there exists $k > 0$ such that $P^k(U) \cap V \neq \emptyset$

2-dimensional maps $F:\mathbb{C}^2 \mapsto \mathbb{C}^2$

$$F : \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} f(z_1, z_2) \\ g(z_1, z_2) \end{pmatrix}$$

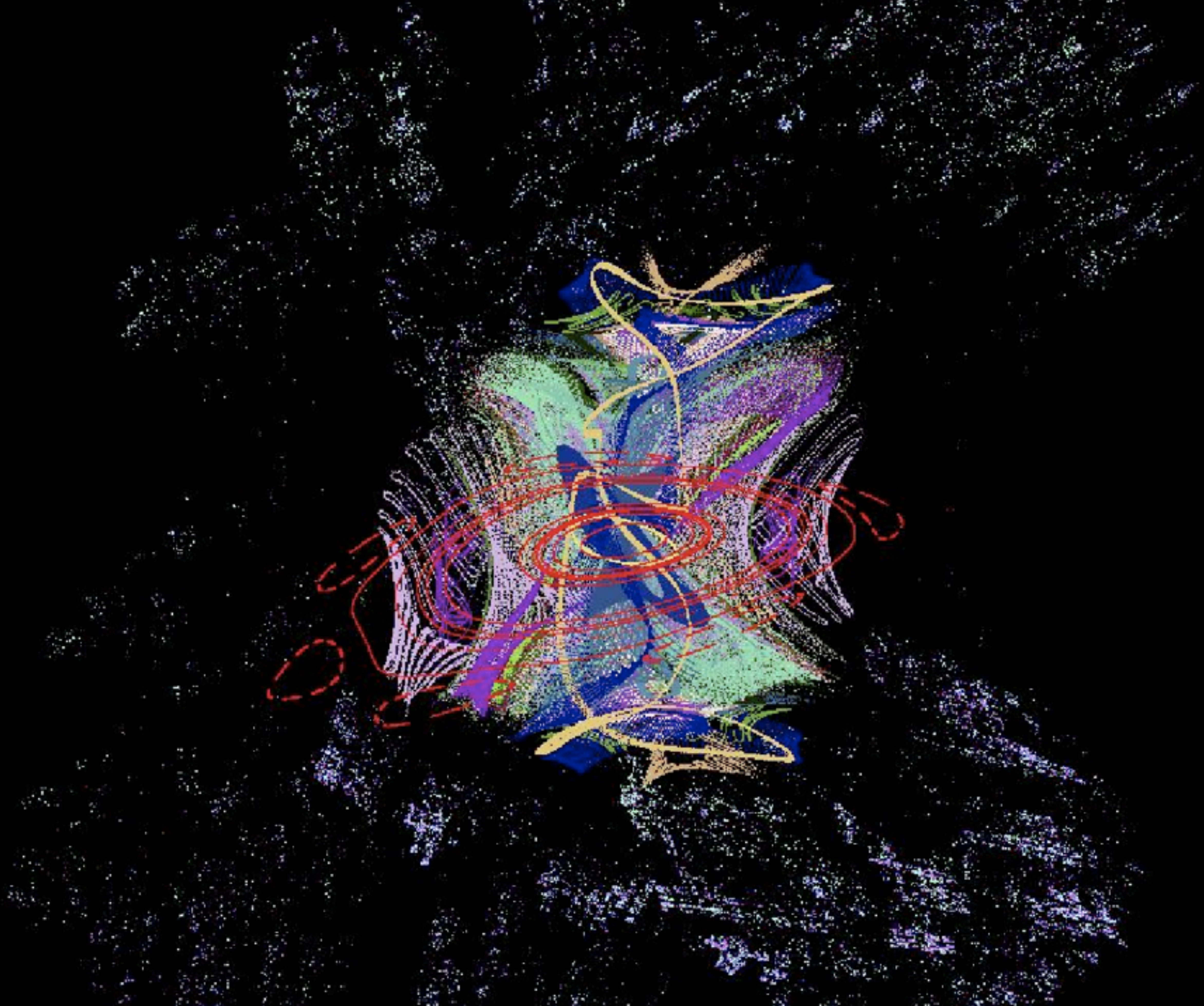
Orbits are classified according to the behavior of $n \to \pm\infty$

$$I^{\pm} = \{ (z_1, z_2) \in \mathbb{C}^2 \mid \lim_{n\to\infty} F^{\pm n}(z_1, z_2) = \infty \}$$

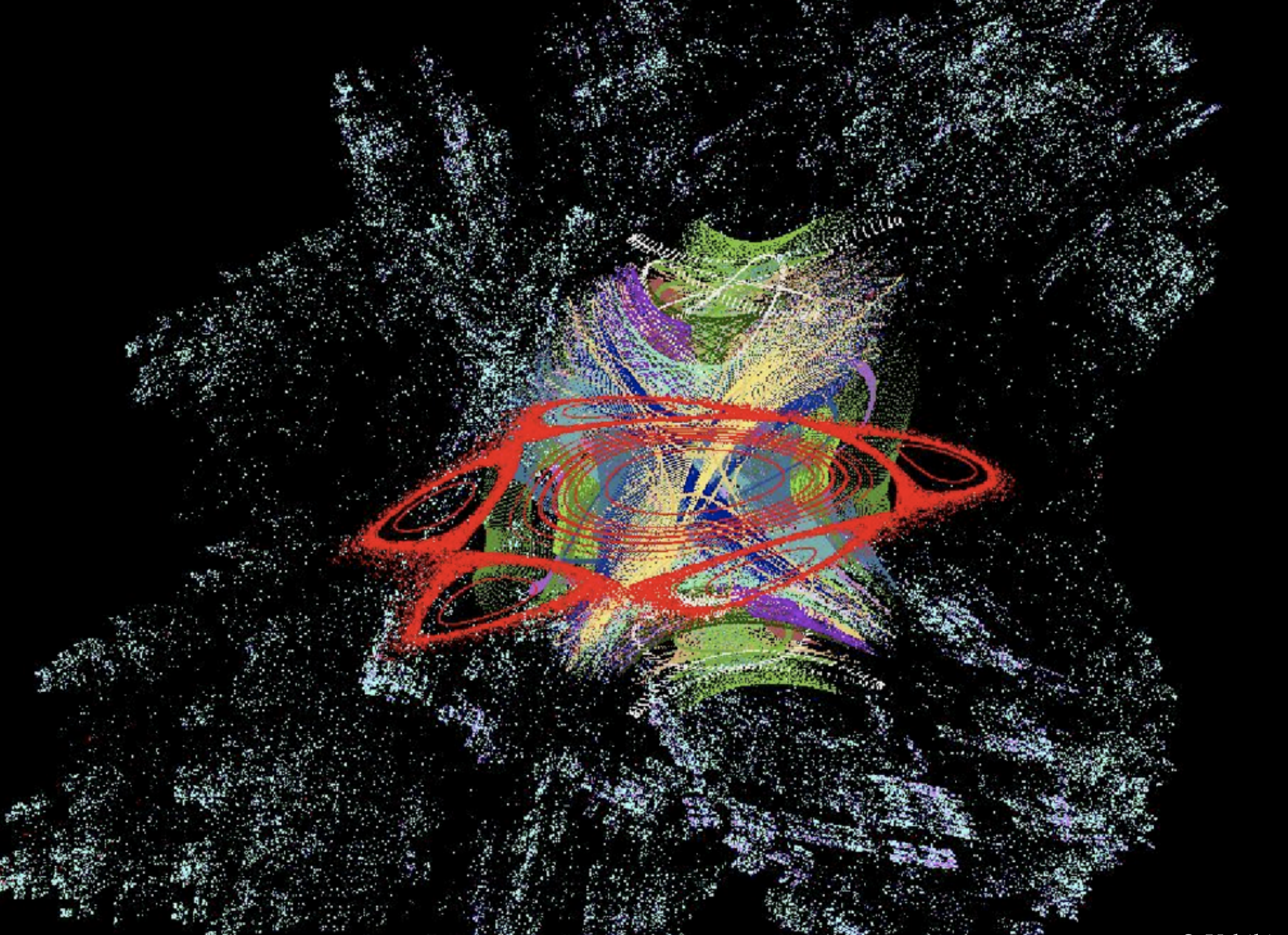
$$K^{\pm} = \{ (z_1, z_2) \in \mathbb{C}^2 \mid \lim_{n\to\infty} F^{\pm n}(z_1, z_2) \text{ is bounded in } \mathbb{C}^2 \}$$

In particular

$K = K^+ \cap K^-$	:	filled Julia set
$J^{\pm} = \partial K^{\pm}$	:	forward (resp. backward) Julia set
$J = J^+ \cap J^-$	:	Julia set



S. Ushiki



S. Ushiki

Green function induced from the dynamics

$$G^{\pm}(z_1, z_2) \equiv \lim_{n\to+\infty} \frac{1}{d^n} \log^{+}\left| F^{\pm n}(z_1, z_2) \right|$$

gives the (1, 1)-currents through the *Poisson equation*

$$\mu^{\pm} \equiv \frac{1}{2\pi} dd^c G^{\pm}$$

Theorem (Bedford-Smillie)

Let M be an algebraic variety, then there is a constant $c > 0$ such that

$$\lim_{n\to\infty} \frac{1}{d^n}[F^{\mp n}M] = c\mu^{\pm}$$

in the sense of current, where $[M]$ is the current of integration of M.

Bedford E and Smillie J :
Invent. Math. **103** (1991) 69-99; *J. Amer. Math. Soc.* **4** (1991) 657-679; *Math. Ann.* **294** (1992) 395-420; *J.Geom.Anal.* **8** (1998) 349-383; *Annal. sci. de l'Ecole norm. super.* **32** (1999) 455-497; *American Journal of Mathematics* **124** (2002) 221-271; *Ann.Math.* **148** (1998) 695-735; *Ann.Math.* **160** (2004) 1-26

Bedford E, Lyubich E and Smillie J
Invent. Math. **112** (1993) 77-125; *Invent. Math.* **114** (1993) 277-288

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in the sense of current, where $[M]$ is the current of integration of M.

Theorem (Bedford-Smillie) $\quad \mathrm{Supp}\ \mu^{\pm} = J^{\pm}$

Theorem (Bedford-Smillie) $\quad$ **F is ergodic on** $J^* = \mathrm{supp}\ (\mu^+ \wedge \mu^-)$

Stable and unstable manifold theorem

$$F : \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p + V'(q) \\ q + p' \end{pmatrix} \qquad (\, V(q) : \text{polynomial}\,)$$

Theorem (Bedford-Smillie 1991) *For any unstable periodic orbits* $\mathbf{p}$,

$$\overline{W^s(\mathbf{p})} = J^+ \quad \textit{and} \quad \overline{W^u(\mathbf{p})} = J^-$$

where $W^s(\mathbf{p})$ (*resp.* $W^u(\mathbf{p})$) *denotes stable* (*resp. unstable*) *manifold for* $\mathbf{p}$ *and* $J^\pm = \partial K^\pm$ *is called the forward* (*backward*) *Julia set*.

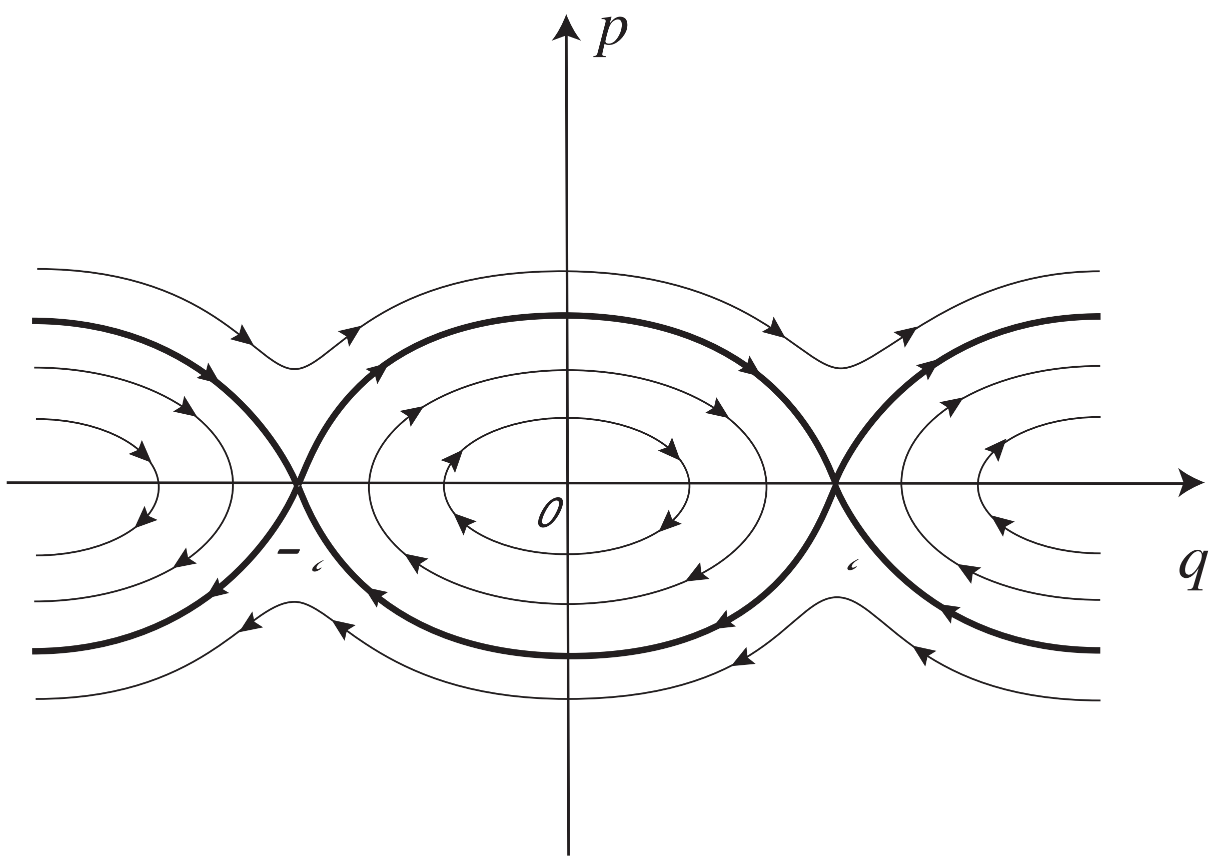
Here, $K^\pm = \{\, (p, q) \in \mathbb{C}^2 \mid \|F^n(p, q)\| \text{ is bounded } (n \to \pm\infty) \,\}$

Note :

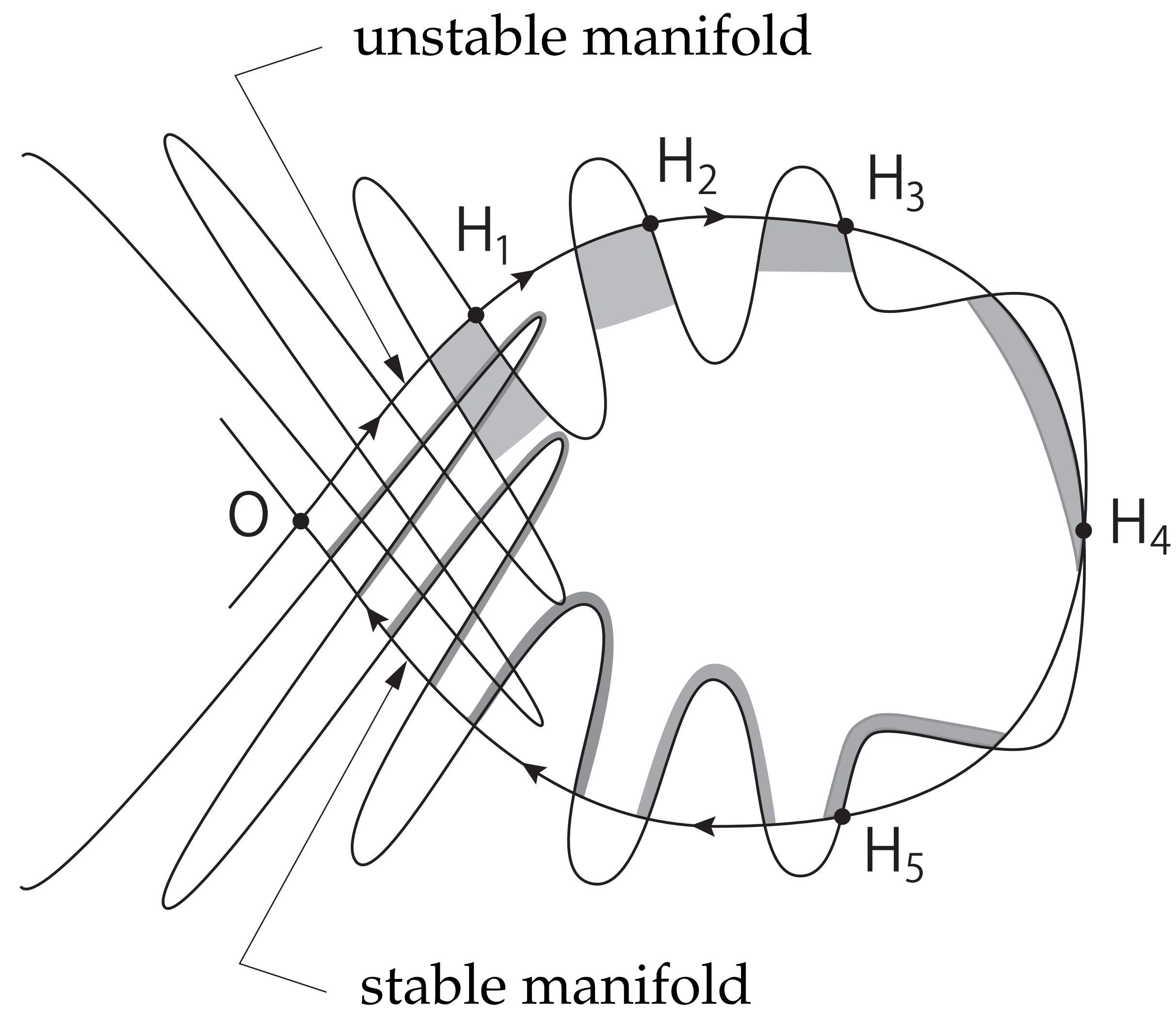
1. This theorem holds even in the system with mixed phase space.
2. $W^s(\mathbf{p})$ and $W^u(\mathbf{p})$ are both locally 1-dimensional complex (= 2-dimensional real) manifold in $\mathbb{C}^2$.

Stable and unstable manifolds in the real phase space

Completely integrable



Nonintegrable



Tunneling orbits and Julia sets

Semiclassical sum

$$K^{sc}(a,b) = \sum_{\gamma} A_n^{(\gamma)}(a,b) \exp\left\{\frac{i}{\hbar} S_n^{(\gamma)}(a,b)\right\}$$

Theorem (AS, Y. Ishii and K.S. Ikeda) **For polynomial maps F,**

(i) If F is hyperbolic and $h_{top}(F|_{\mathbb{R}^2}) = \log 2$, then $C = J^+$

(ii) If F is hyperbolic and $h_{top}(F|_{\mathbb{R}^2}) > 0$, then $\overline{C} = J^+$

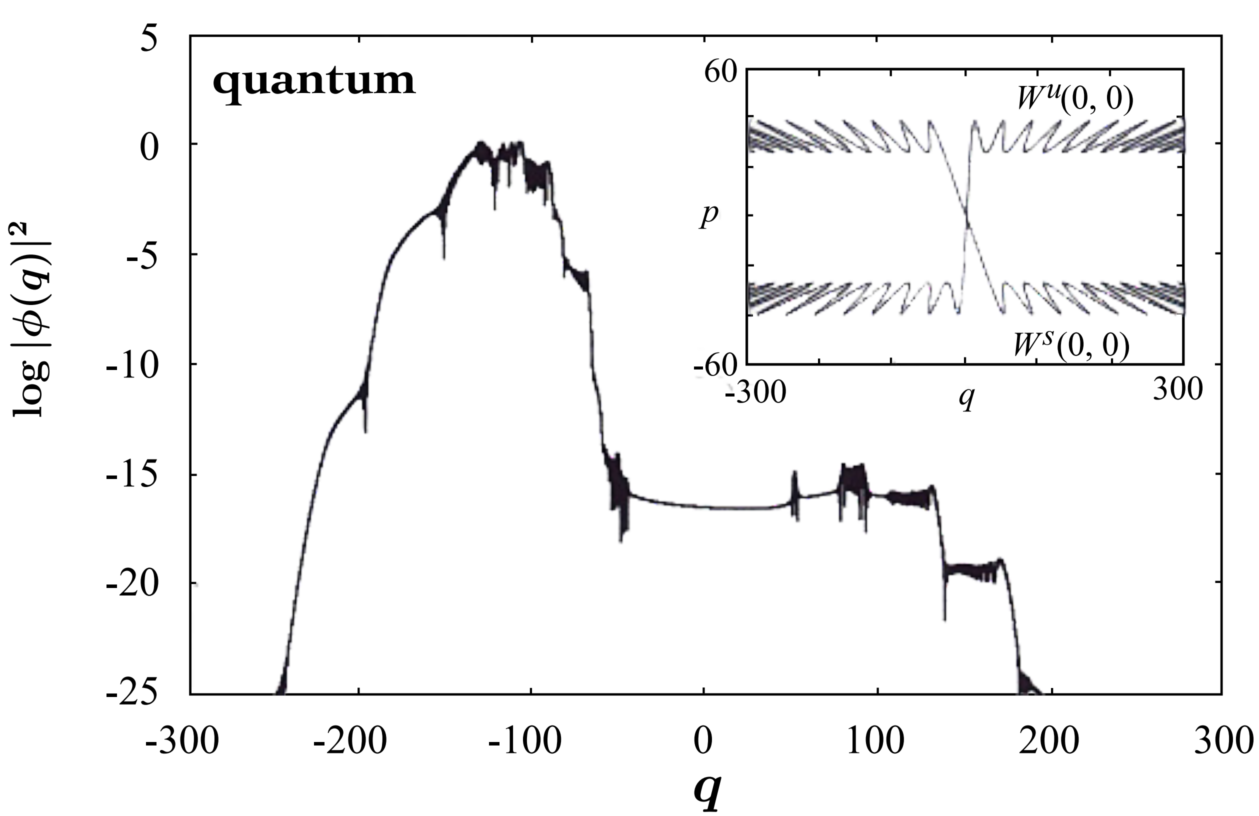
(iii) If $h_{top}(F|_{\mathbb{R}^2}) > 0$, then $J^+ \subset \overline{C} \subset K^+$

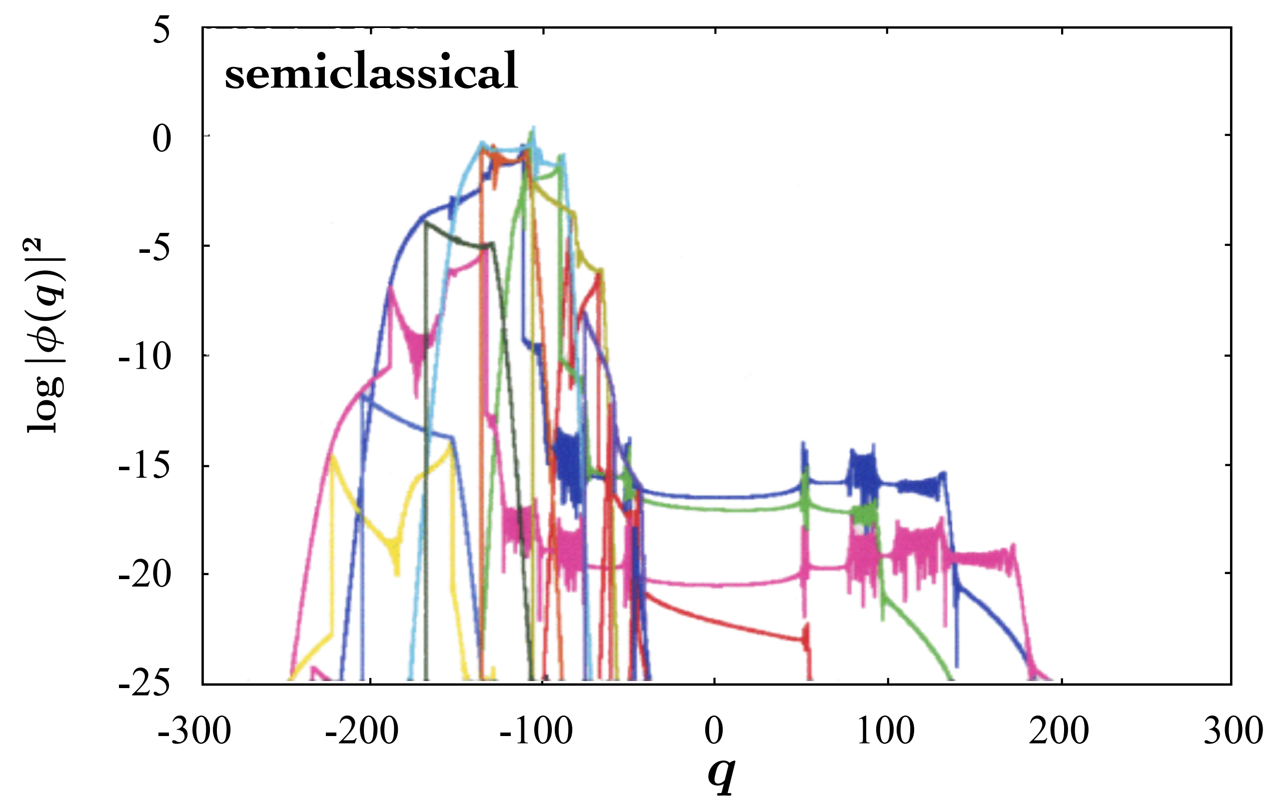
Here $h_{top}(P|_{\mathbb{R}^2})$ is topological entropy confined on $\mathbb{R}^2$, and semiclassically contributing complex orbits are introduced as

$$C \equiv \{ (q,p) \in \mathcal{M}_\infty \mid \text{Im}\, S_n(q,p) \text{ converges absolutely at } (q,p) \}$$

(Proof) apply the convergent theory of current (Bedford-Smillie)

Comparison between quantum and semiclassical (numerics)





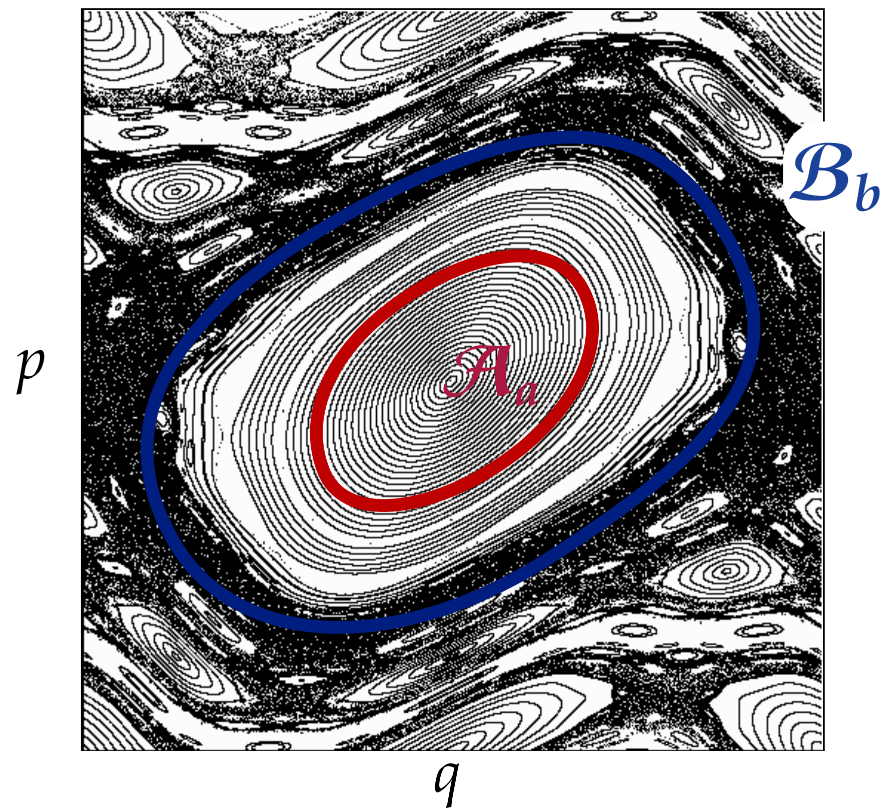
Not all of complex paths contribute ...

Quantum propagator:

$$K(a,b) = \langle b|\hat{U}|a\rangle = \int \cdots \int \prod_j dq_j \prod_j \exp\left[\frac{\mathrm{i}}{\hbar} S(\{q_j\},\{p_j\})\right]$$

Semiclassical approximation of propagator

$$K^{\mathrm{sc}}(a,b) = \sum_\gamma A_n^{(\gamma)}(a,b) \exp\left[\frac{\mathrm{i}}{\hbar} S_n^{(\gamma)}(a,b)\right]$$



Evaluation of integrals with a large (small) parameter

Integral (single, multiple, infinite) with a large parameter η:

$$I(\eta, \mathcal{X}) = \int \cdots \int_C g(z_1, \cdots, z_N) \exp\Big[\eta S(z_1, \cdots, z_N; \mathcal{X})\Big] dz_1 \cdots dz_N$$

where $\mathcal{X} = (x, y, z, \cdots)$ is a set of parameters.

$\boldsymbol{I(\eta, \mathcal{X})}$ can be Feynman path integrals in quantum mechanics, partition functions in field theory, diffraction integrals in optics $\cdots$,

For simplicity,

$$I(\eta, \mathcal{X}) = \int_C \exp\Big[\eta S(z; \mathcal{X})\Big] dz$$

To evaluate $\boldsymbol{I(\eta, \mathcal{X})}$, saddle-point (stationary phase) approximation is efficient and often used.

Conventional saddle point method

- Saddle point method had been used only as a tool to evaluate integrals approximately.
- Remainders $R_N^{(i)}$ had been regarded as uncontrollable errors without meaningful information.

$$I(\eta, \mathcal{X}) = \sum_i \exp\left[\eta S(z_i; \mathcal{X})\right]\left(\sum_{r=0}^{N-1} A_r^{(i)} \eta^{-r} + R_N^{(i)}\right)$$

However,

Remember that expansions are divergent because there exist multiple saddles.

In other words, the convergence of the expansion around a saddle is prevented by other saddles.

Expansions around different saddles might be related with each other.

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Resurgent theory

Asymptotic expansion around saddle z_i

$$I(\eta, \mathcal{X}) = \sum_i \exp\Big[\eta S(z_i; \mathcal{X})\Big] \mathcal{I}^{(i)} \quad \text{where} \quad \mathcal{I}^{(i)} = \sum_{r=0}^{N-1} A_r^{(i)} \eta^{-r} + R_N^{(i)}$$

Remainder term $R_N^{(i)}$ can be expanded around the other saddles z_j (Berry-Howls 1991)

$$R_N^{(i)} = \frac{1}{2\pi \mathrm{i}} \sum_j \left(\frac{1}{\eta S_{ij}}\right)^N \int_0^\infty dz \frac{\mathrm{e}^{-z} z^{N-1}}{1 - z/(\eta S_{ij})} \mathcal{I}^{(j)}\left(\frac{z}{S_{ij}}\right)$$

where $S_{ij} \equiv S(z_j; \mathcal{X}) - S(z_i; \mathcal{X})$.

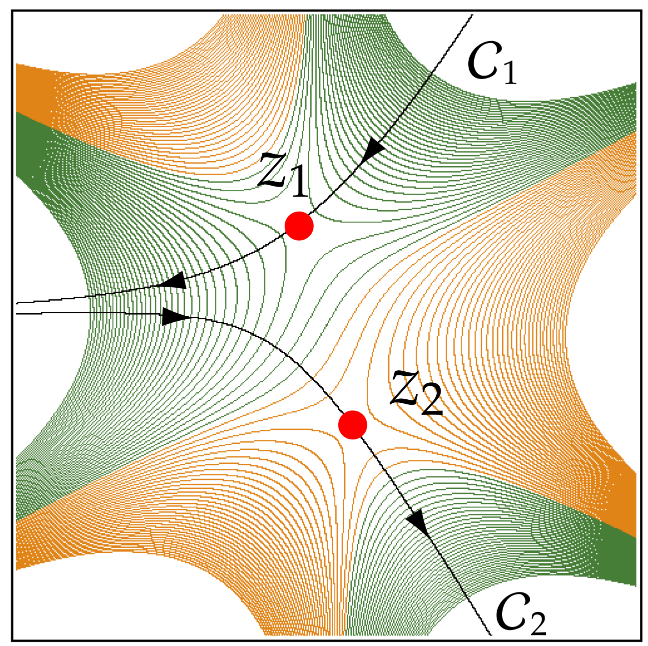
- Information for the asymptotic series around the saddles $z_j, z_k, \cdots$ is contained in the remainder term of the saddle z_i.

- Each asymptotic series communicates with others through remainder terms.

Saddle point method and Stokes phenomenon

Saddle points z_i are points satisfying $\left.\dfrac{\partial S(z,\mathcal{X})}{\partial z}\right|_{z=z_i}$

Steepest descent curves C_i **associated with** z_i are contour curves of $\mathrm{Im}\, S(z,\mathcal{X})$ = const passing through the saddle points z_i



Decompose the integral into a sum over saddles

$$I(\eta,\mathcal{X}) \;=\; \sum_i \int_{C_i} \exp\Big[\eta S(z;\mathcal{X})\Big] dz$$

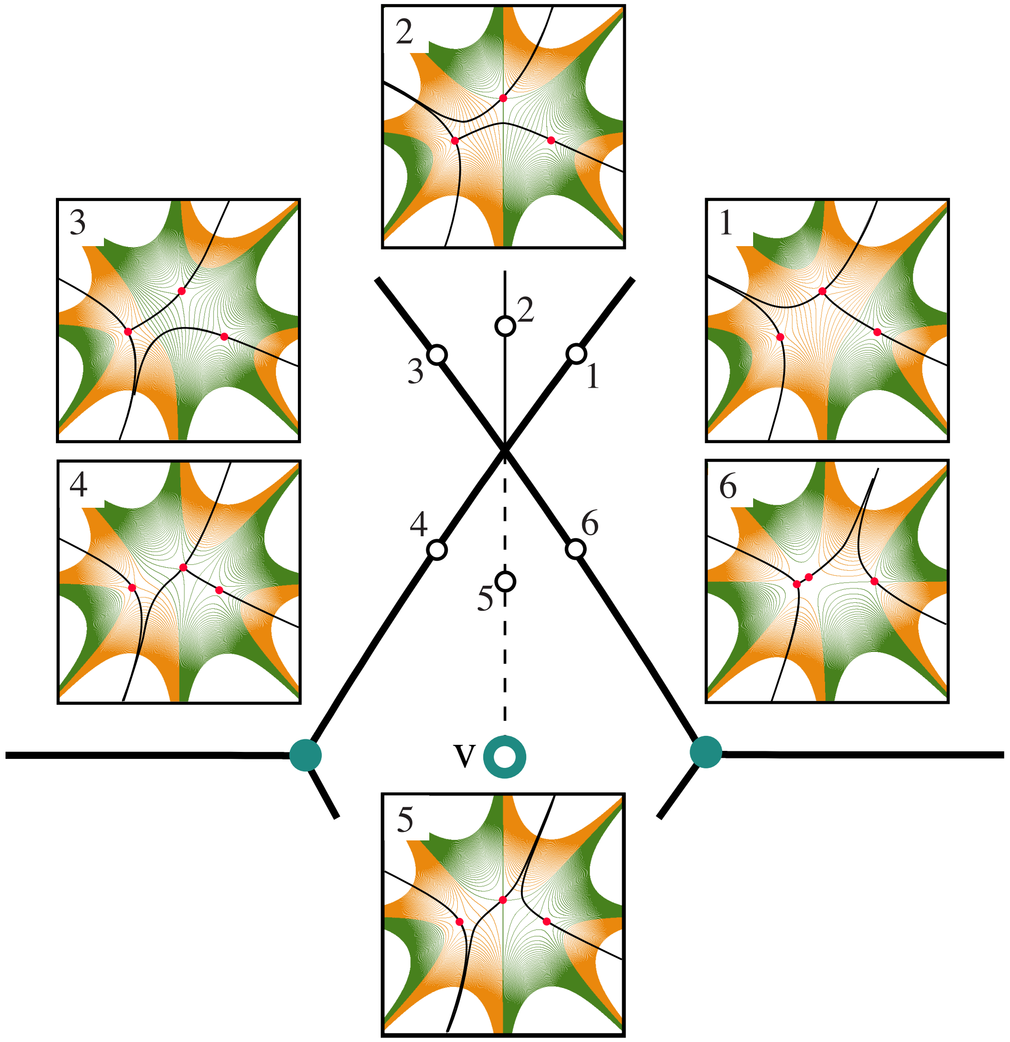
Stokes phenomenon in case with more than two saddles

3rd order differential equations

$$\left(\eta^{-3}\frac{d^3}{dz^3} + 3\eta^{-1}\frac{d}{dz} + iz\right)\varphi = 0 \qquad (\eta\text{: large parameter})$$

- **Necessity to introduce *new Stokes curves***
  (Berk, Nevins and Roberts, 1982)

- ***Virtual turning points* and exact WKB foundation for higher-order differential equations**
  (Aoki, Kawai and Takei, 1994)



***Virtual turning points and new Stokes curves*:**

Any similar, or even related, precedents do not exist in the traditional asymptotic analysis

Stokes phenomenon for multistep quantum propagator

n-step quantum propagator for the Hénon map

$$u(q_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dq_1 dq_2 \cdots dq_{n-1} \exp\Big[\frac{i}{\hbar} S(q_0, q_1, \cdots, q_n)\Big]$$

where

$$S(q_0, q_1, \cdots, q_n) = \sum_{j=1}^{n} \frac{1}{2}(q_j - q_{j-1})^2 - \sum_{j=1}^{n-1} V(q_j)$$

and

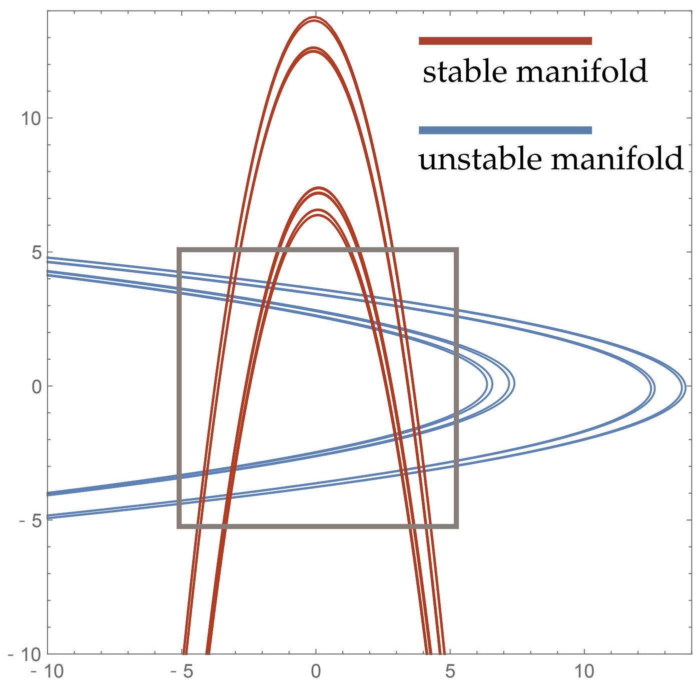
$$V(q) = -\frac{1}{3}q^3 + cq$$

Saddle point condition

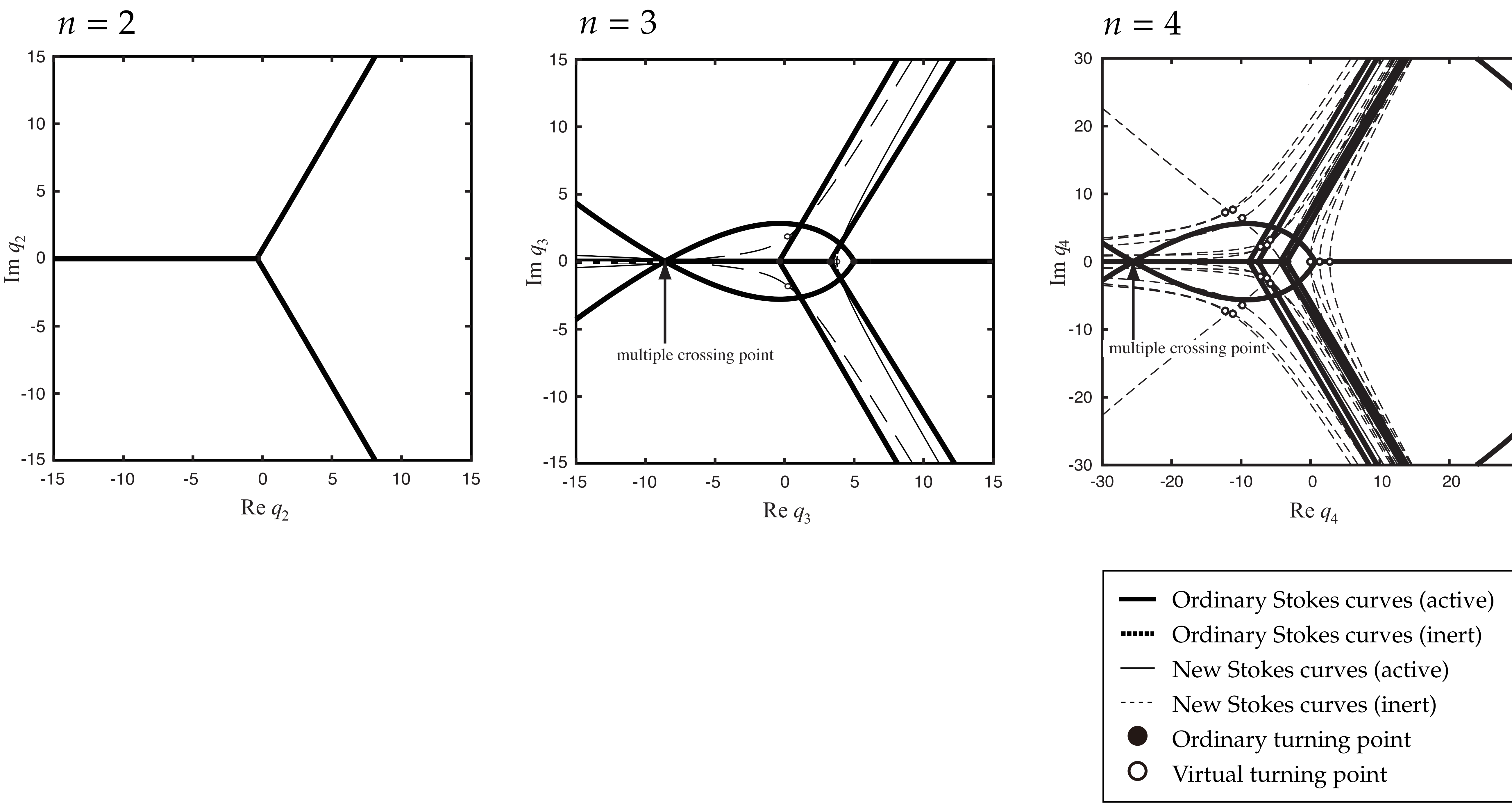
$$\frac{\partial}{\partial q_i} S(q_0, q_1, \cdots, q_n) = 0 \qquad (1 \leq i \leq n-1)$$

leads to the area-preserving Hénon map

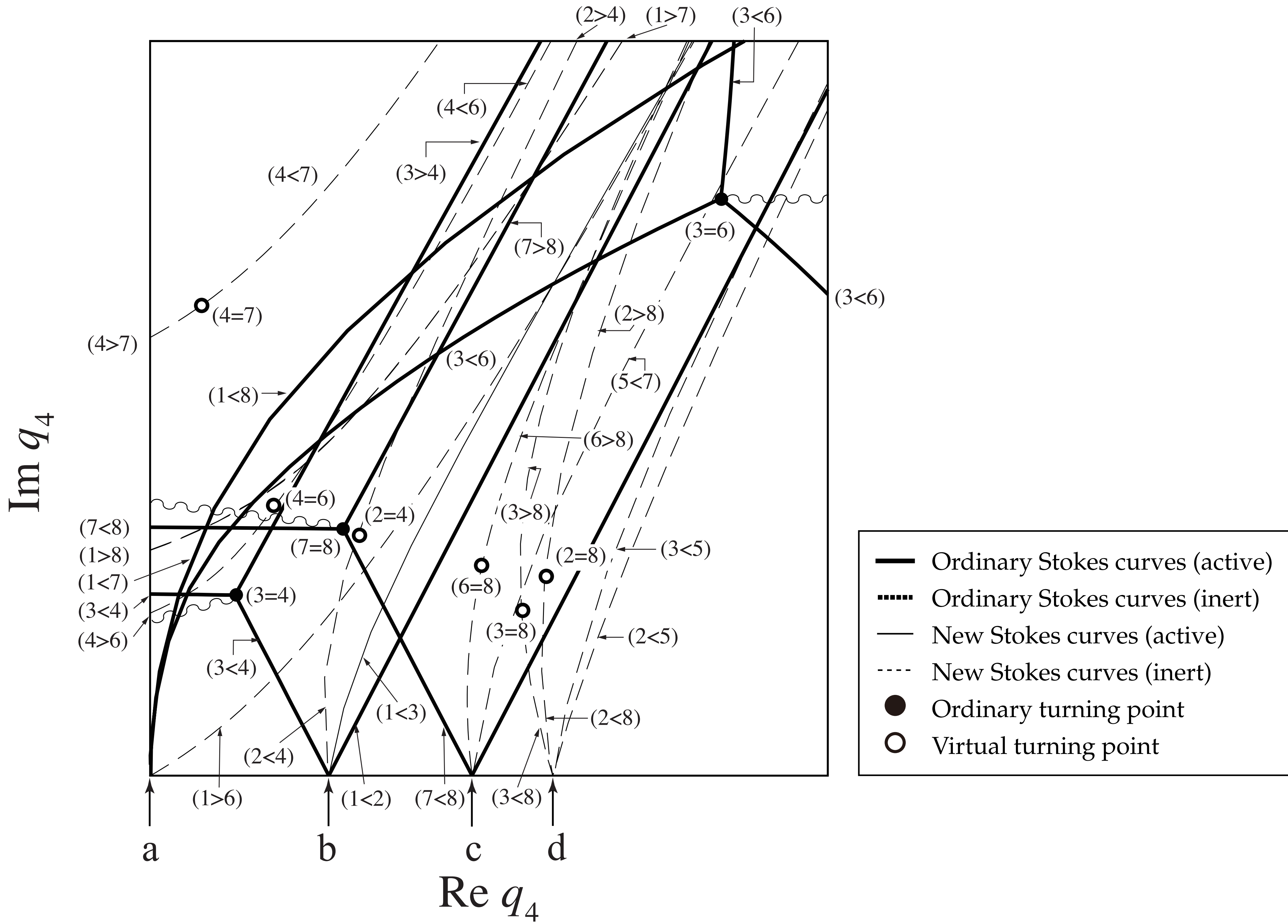
$$F : \begin{pmatrix} p_{i+1} \\ q_{i+1} \end{pmatrix} = \begin{pmatrix} p_i - V'(q_i) \\ q_i + p_{i+1} \end{pmatrix}$$



Time evolution of Stokes geometry



Stokes geometry in a generic situation



Summary

- Signature of quantum tunneling drastically changes due to the presence of chaos.

- In nonintegrable systems, classically disconnected regions are connected via the orbits in the Julia set.

- Strong enhancement of tunneling probability occurs because of an abundance of complex orbits.

- Stokes phenomenon in nonintegrable systems is a challenging issue, and resurgent theory play a crucial role there.