Inflationary Universe

1. Problems in Standard Model

1.1 Flatness Problem

Friedmann Equation
$$\left(\frac{\dot a}{a}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

$$\begin{cases} H &= \frac{\dot a}{a} \\ \Omega &= \frac{8\pi G\rho}{3H^2} \\ \lambda &= \frac{\Lambda}{3H^2} \end{cases} \quad \Rightarrow \quad H^2 = \Omega H^2 - \frac{K}{a^2} + \lambda H^2$$

$$\boxed{\Omega + \lambda - 1 = \frac{K}{a^2 H^2}}$$

$T_{eq} \sim 1\mathrm{eV} = 10^{-9}\mathrm{GeV}$

$$\Omega + \lambda - 1 = \frac{K}{a^2 H^2} \propto \begin{cases} T^{-2} & (RD) \quad a \propto 1/T, \quad H \propto \rho^{1/2} \propto T^2 \\ T^{-1} & (MD) \quad a \propto 1/T, \quad H \propto \rho^{1/2} \propto T^{3/2} \end{cases}$$

At present

$$\Omega_0 \sim O(1) \quad \lambda \sim O(1) \qquad T_0 \simeq 2.7\mathrm{K} \simeq 3 \times 10^{-4}\mathrm{eV}$$

At Planck time $T \sim 10^{19}\mathrm{GeV}$

$$\Omega + \lambda - 1 \sim \left(\frac{10^{19}}{10^{-9}}\right)^{-2} \left(\frac{10^{-9}}{10^{-13}}\right)^{-1} \sim 10^{-60}$$

$\Rightarrow$ $\Omega + \lambda = 1$ with accuracy 10^{-60}

unnatural fine tuning !

$\Rightarrow$ $\boxed{\text{Flatness Problem}}$

Entropy of the Universe

$$S = a^3 s \quad \Rightarrow \quad a = (S/s)^{1/3}$$

$$\boxed{S = \left[\frac{K}{H^2(\Omega + \lambda - 1)}\right]^{3/2} s}$$

$K = 0, \pm 1$ $\quad {}^{(3)}R = \pm 1/a^2$

Re-definition of r, a

$$\boxed{S = \left[\frac{K}{H^2(\Omega + \lambda - 1)}\right]^{3/2} s}$$

At Present

$$\begin{aligned} s_0 &= \frac{2\pi}{45}\left(2T^3_{\gamma,0} + 2\frac{7}{8}3T^3_{\nu,0}\right) = \frac{2\pi}{45}\left(2T^3_{\gamma,0} + 2\frac{7}{8}3\frac{4}{11}T^3_{\gamma,0}\right) \\ &= 1.715T^3_{\gamma,0} = 2.8 \times 10^3\ \mathrm{cm}^{-3} \end{aligned}$$

$$H_0^{-1} \simeq 3000\ \mathrm{Mpc} \sim 10^{28}\ \mathrm{cm}$$

$\Rightarrow$ $S \gtrsim 10^{87}$

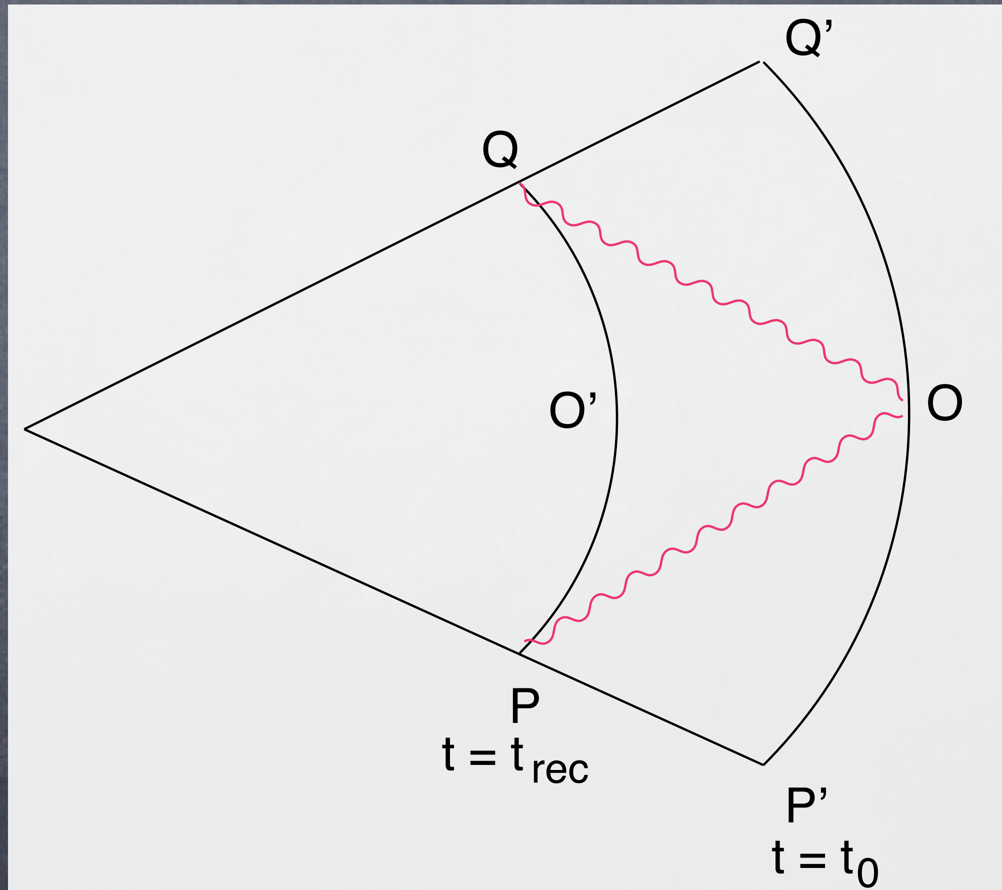
$\boxed{\text{Large Entropy Problem}}$

1.2 Horizon Problem

CMB is extremely isotropic

emitted at recombination (T ~ 3000K)

Observer at present sees CMB coming from two opposite directions



distance between P and O

$$PO'(t_{rec}) = a(t_{rec}) \int_{t_{rec}}^{t_0} \frac{dt'}{a(t')}$$

$$= bt^{2/3} \int_{t_{rec}}^{t_0} \frac{dt'}{bt'^{2/3}}$$

$$= 3(t_{rec}^{2/3} t_0^{1/3} - t_{rec})$$

$$(a \propto t^{2/3} \quad MD)$$

Particle horizon at recombination

$$\ell_H(t_{rec}) = bt_{rec}^{2/3}\int_0^{t_{rec}} \frac{dt'}{bt'^{2/3}} = 3t_{rec}$$

$$N = \frac{PQ}{\ell_H} = 2\left[\left(\frac{t_0}{t_{rec}}\right)^{1/3} - 1\right]$$
$$= 2\left[\left(\frac{T_{rec}}{T_0}\right)^{1/2} - 1\right] = 2\left[\left(\frac{3000\mathrm{K}}{2.7\mathrm{K}}\right)^{1/2} - 1\right] \simeq 74$$

⇒ P and Q are far away and have no causal relation

unnatural ⇒ $\boxed{\text{Horizon Problem}}$

1.3 Monopole Problem

GUTs (Grand Unified Theories)

Interaction among elementary particles can be described by one gauge interaction with symmetry represented by group G

$$G \Rightarrow SU_c(3) \times SU_L(2) \times U_Y(1)$$

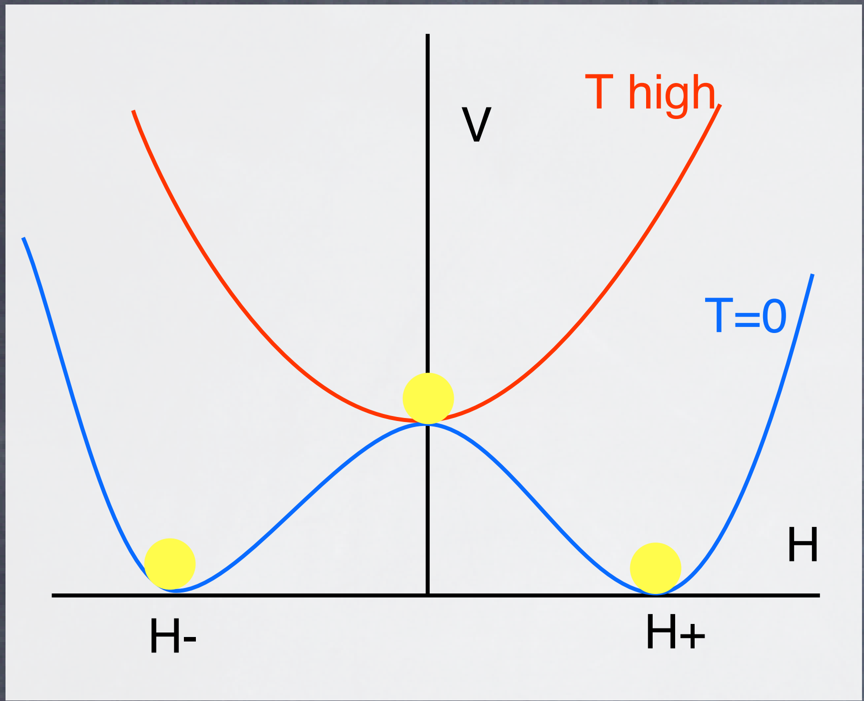
$\uparrow$

Higgs Mechanism   H: Higgs field $\langle H \rangle = 0 \rightarrow \langle H \rangle \neq 0$

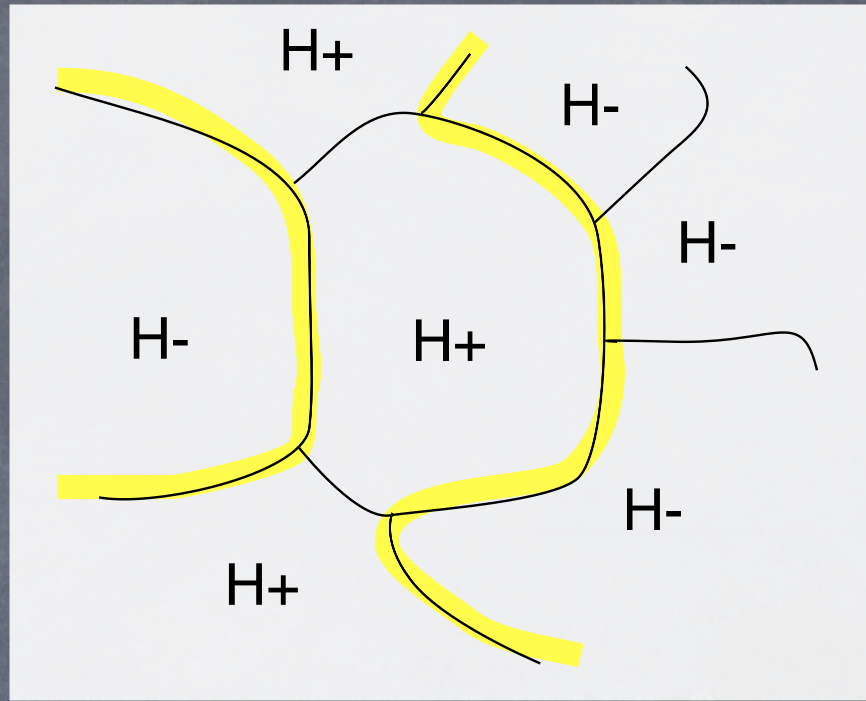
When symmetry G is spontaneously broken to U(1) , topological defects (monopoles) are produced (Kibble Mechanism)

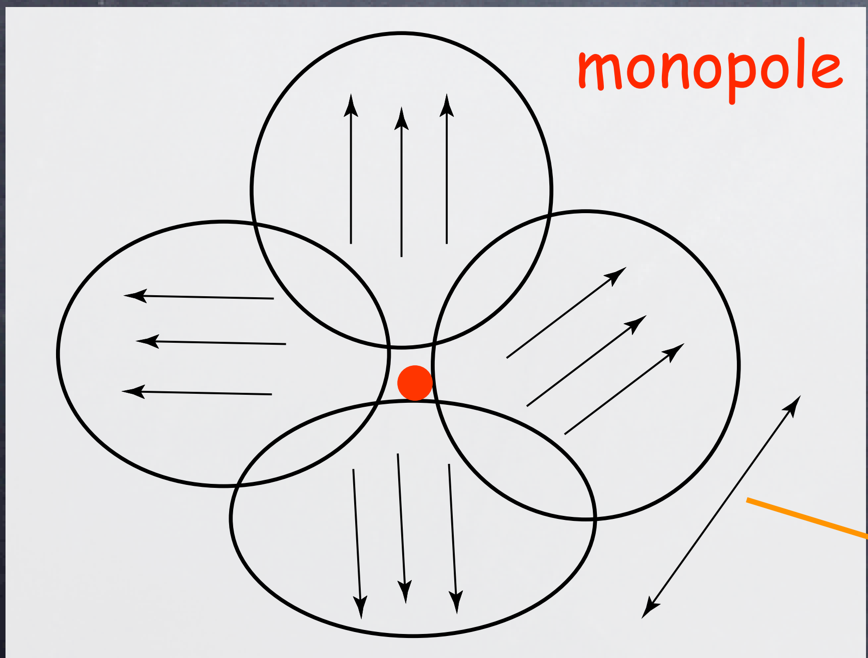
Topological Defect
- Domain Wall (2dim)
- Cosmic String (1dim)
- Monopole (0dim)

Example (Domain Wall)



Domain Wall





coherent length

Monopole density at formation epoch t_f

$$\text{coherent length } \xi < \ell_H = 2t_f$$

$$\Rightarrow \; n_M = \xi^{-3} \geq \ell_H^{-3} = \frac{1}{8t_f^3} \qquad \leftarrow \quad \propto a^{-3}$$

Entropy density

$$s = \frac{2\pi^2}{45} g_* T_f^3 \qquad \leftarrow \quad \propto a^{-3}$$

$$\frac{n_M}{s} \geq \frac{1}{8t_f^3}\left(\frac{2\pi^2}{45} g_* T_f^3\right)^{-1} = \frac{1}{8}\left(\frac{2\pi^2 g_*}{45}\right)^{3/2} \frac{T_f^6}{M_G^3}\left(\frac{2\pi^2}{45} g_* T_f^3\right)^{-1}$$

$$= \frac{\pi}{4\sqrt{2}\sqrt{45}} g_*^{1/2} \frac{T_f^3}{M_G^3} \simeq 0.8\left(\frac{T_f}{M_G}\right)^3$$

$$T_f = 10^{15} \text{ GeV} \quad \Rightarrow \; \frac{n_M}{s} \geq 0.8\left(\frac{10^{15}}{2.4\times 10^{18}}\right)^3 \simeq 6\times 10^{-11}$$

$$\left(\frac{n_M}{s}\right)_0 = \left(\frac{n_M}{s}\right)_f \qquad s_0 = 2.8 \times 10^3 \ \mathrm{cm}^{-3}$$

Monopole mass

$$m_M \simeq 10^{16} \ \mathrm{GeV}$$

Monopole mass density

$$\rho_M \geq 1.6 \times 10^9 \ \mathrm{GeVcm}^{-3}$$

$$\rho_c = 1.053 \times 10^{-5} h^2 \ \mathrm{GeVcm}^{-3}$$

$$\boxed{\Omega_M \geq 1.6 \times 10^{14} \gg 1}$$

$\Rightarrow$ Monopole Problem

1.4 Gravitino Problem

Supersymmetry (SUSY)

Fermion $\longleftrightarrow$ Boson

- Hierarchy Problem
  Keep electroweak scale against radiative correction
- Coupling Constant Unification in GUT

quark $\longleftrightarrow$ squarks
lepton $\longleftrightarrow$ slepton
photon $\longleftrightarrow$ photino

$\boxed{\text{Gravitino } \psi_{3/2}}$ Superpartner of graviton

mass of gravitino ~ O(100) GeV

Gravitinos are expected to be in thermal equilibrium at Planck time

$$\Rightarrow \quad n_{3/2} \sim n_\gamma$$

Lifetime of gravitino

$$\tau(\psi_{3/2} \to \tilde{\gamma} + \gamma) \simeq 4 \times 10^8 \,\mathrm{sec} \left(\frac{m_{3/2}}{100\mathrm{GeV}} \right)^{-3}$$

Entropy production

$$\frac{\rho_{3/2}}{\rho_\gamma} \sim \frac{m_{3/2} n_{3/2}}{T n_\gamma} \sim \frac{m_{3/2}}{T} \gg 1 \qquad \left(\leftarrow \begin{array}{c} t \sim 10^8 \,\mathrm{sec} \\ T \sim 100 \,\mathrm{eV} \end{array} \right)$$

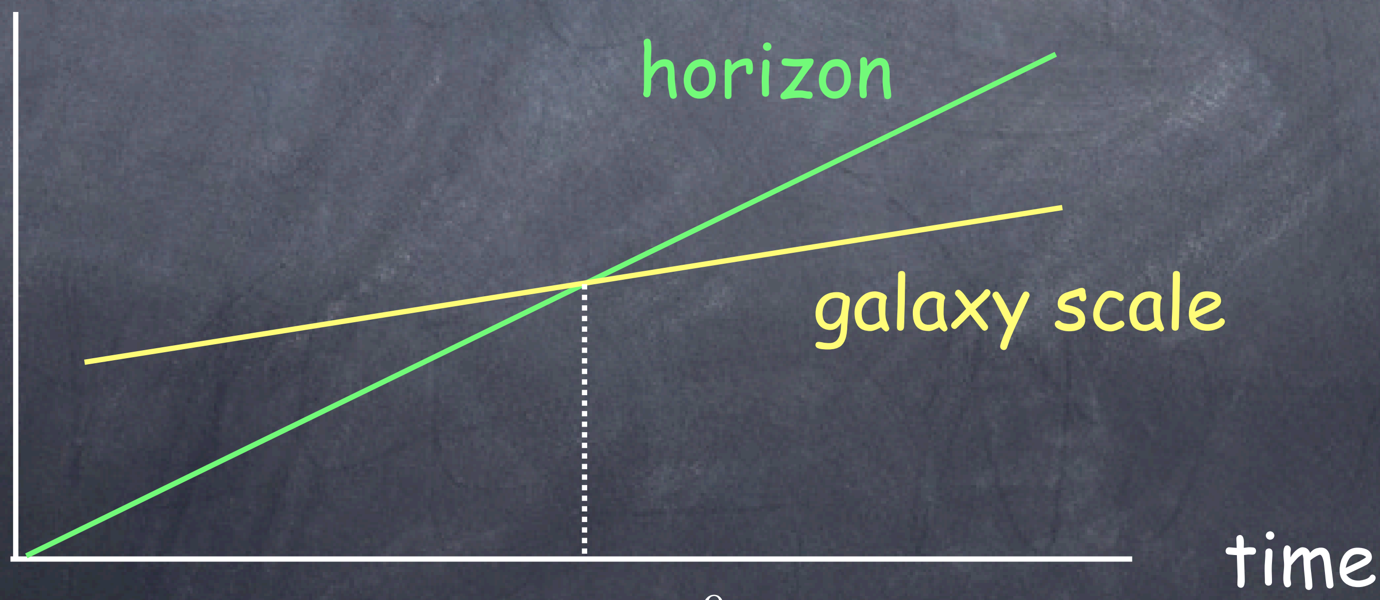
⇨ Huge entropy production

⇨ Too much dilute baryon density $\quad \mathrm{BBN} \Rightarrow \dfrac{n_B}{n_\gamma} \sim 10^{-10}$

$\boxed{\text{Gravitino Problem}}$

1.5 Density Fluctuation Problem

- Structures of the Universe (galaxies...) are formed from small density fluctuations through gravitational instability
- However, no mechanism to produce the fluctuations in the standard model is found



$$t_{gal} \simeq 2 \times 10^9 \,\mathrm{sec}$$

2. Success of Inflation Model

2.1 Inflationary Universe

For some reason vacuum energy dominates the universe

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho_v = H^2 \qquad \rho_v : \text{ vacuum energy} \sim \text{const}$$

$$\Rightarrow \begin{cases} a(t) & \propto & \exp(Ht) \\ H & = & \left[\frac{8\pi G}{3}\rho_v\right]^{1/2} \end{cases}$$

Exponential expansion

Inflation Model

- Exponential (accelerated) expansion
- End of exponential expansion and release of vacuum energy into radiation

2.2 Flatness Problem

During inflation the scale factor increases by

$$\boxed{\frac{a_f}{a_i} = \exp[H(t_f - t_i)] \equiv Z}$$

RD | inflation | RD

t_i t_f

T_R reheating temp

$$t = t_i \Rightarrow \quad \rho_{R,i} \sim \rho_v$$

$$t = t_f \Rightarrow \quad \rho_{R,f} \sim \rho_v$$

$\Rightarrow$ $\boxed{T_i = T_R}$

$\Rightarrow$ entropy density $\quad s(t_i) \ = \ s(t_f)$

scale factor $\quad Za(t_i) = a(t_f)$

total entropy $\quad Z^3 S(t_i) \ = \ S(t_f)$

$$Z \gtrsim 10^{29} \quad \Rightarrow \quad S(t_f) \gtrsim 10^{87}$$

even if S(ti)~1

2.3 Horizon Problem

Horizon length just before inflation

$$\ell_H(t_i) \simeq 2t_i$$

Just after inflation

$$\ell_H(t_f) \simeq 2t_i Z$$

At present the scale corresponding the above horizon

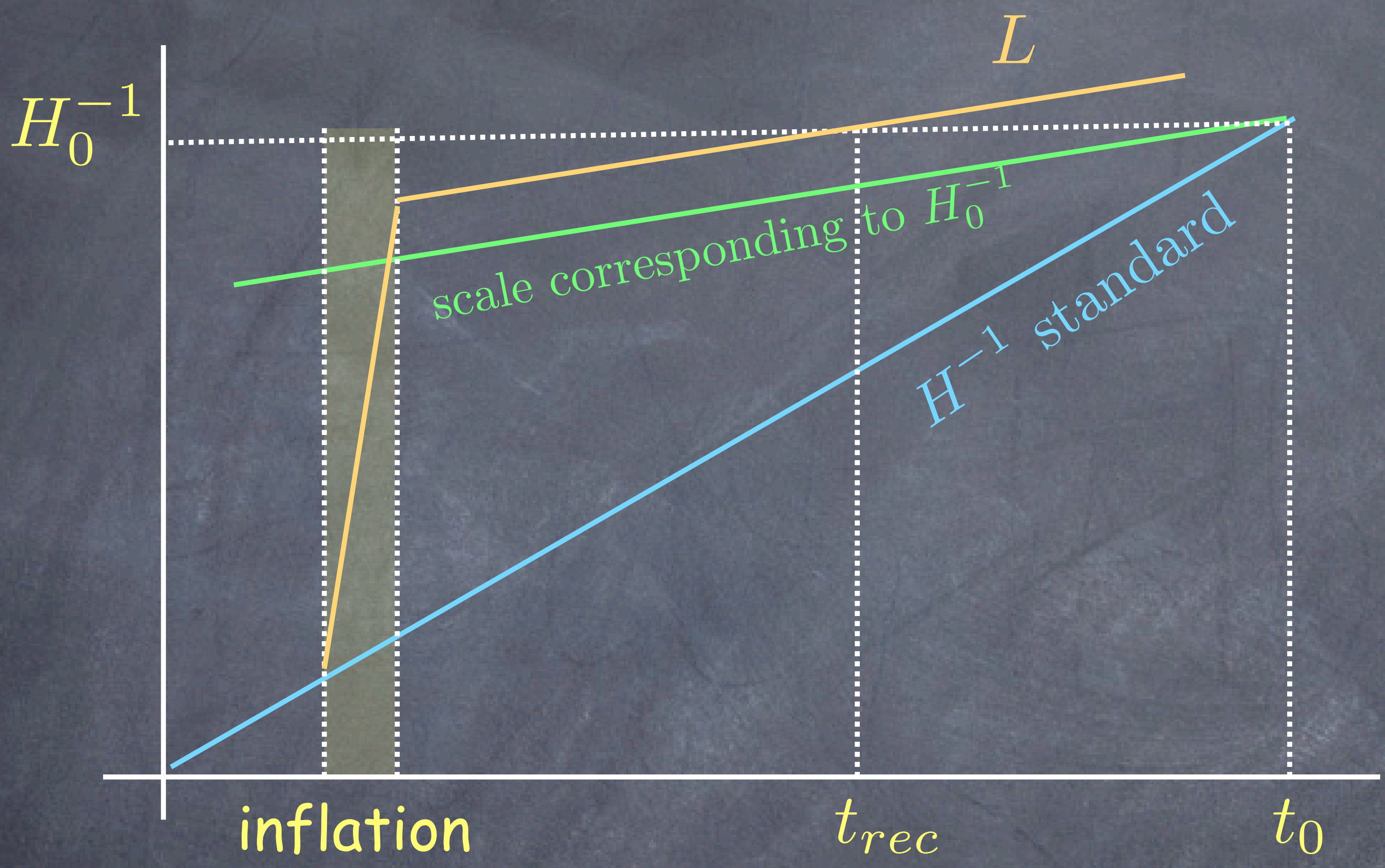
$$L \simeq \frac{a(t_0)}{a(t_f)} 2t_i Z$$

$$t_i \sim H^{-1} \sim \frac{M_G}{\rho_v^{1/2}} \sim \frac{10^{18}\mathrm{GeV}}{(10^{16}\mathrm{GeV})^2} \simeq (10^{14}\mathrm{GeV})^{-1} \simeq 2\times 10^{-28}\mathrm{cm}$$

$$T_R \sim 10^{16}\mathrm{GeV} \;\rightarrow\; \frac{a(t_0)}{a(t_f)} \sim \left(\frac{10^{-4}\mathrm{eV}}{10^{16}\mathrm{GeV}}\right)^{-1} \sim 10^{29}$$

$\Rightarrow$ $L \sim 40Z \text{ cm} \gtrsim H_0^{-1} \sim 10^{28} \text{ cm}$ $\Rightarrow$ $Z \gtrsim 10^{26}$

$(H\Delta t \gtrsim 66)$



L
H_0^{-1}
scale corresponding to H_0^{-1}
H^{-1} standard
inflation
t_{rec}
t_0

2.5 Other Problems

Monopole

Gravitino

Diluted by $Z^{-3} \gtrsim 10^{-87}$

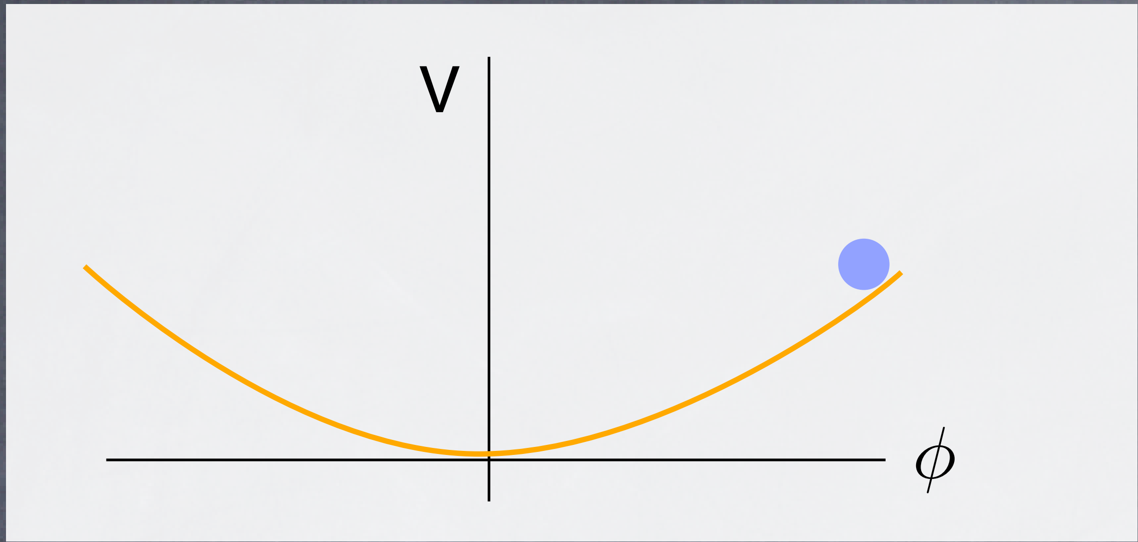
Density

Fluctuations

Inflation has mechanism

generating density fluctuations

3. Chaotic Inflation Model



$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$$

$$V(\phi) = \frac{1}{2} m^2 \phi^2$$

In general $V(\phi) = \dfrac{\lambda \phi^n}{n M_G^{n-4}}$ $\quad 0 < \lambda \ll 1$

3.1 Chaotic Condition

Initial Condition? $t = t_{pl}$

Heisenberg Uncertainty Principle $\Delta E \, \Delta t \gtrsim 1$

Energy density

$$\Delta\rho \sim \Delta E/(\Delta x)^3 \gtrsim 1/(\Delta t)(\Delta x)^3 \gtrsim M_{pl}^4$$

$$\Delta x \lesssim \ell_H \sim M_{pl}^{-1}$$

$$\Delta t \lesssim t_{pl} \sim M_{pl}^{-1}$$

$\Rightarrow$ Energy density is determined only with accuracy $O(M_{pl})$

$$\Rightarrow \begin{cases} \partial_0\phi\partial_0\phi & \sim & M_{pl}^4 \\ \partial_i\phi\partial_i\phi & \sim & M_{pl}^4 \\ V(\phi) & \sim & M_{pl}^4 \end{cases}$$

$\Leftarrow$ Chaotic Condition of the Early Universe

3.2 Dynamics of Inflaton

initial value of Φ is large

$$m^2\phi_i^2 \sim M_{pl}^4 \quad m \ll M_{pl} \;\Rightarrow\; \phi_i \sim M_{pl}^2/m \gg M_{pl}$$

Suppose, in some region

$$(\partial_0\phi)^2, (\partial_i\phi)^2 \; < \; V(\phi) \sim M_{pl}^4$$

Friedmann eq.

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_G^2}\left(\frac{\dot{\phi}^2}{2} + \frac{(\nabla\phi)^2}{2a^2} + V(\phi)\right)}$$

Evolution of inflaton

$$S = \int \sqrt{-g}\mathcal{L}d^4x = \int a^3\mathcal{L}d^4x$$

$$\boxed{\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\triangle\phi = -\frac{dV}{d\phi}}$$

$$\dot{\phi}^2, (\nabla\phi)^2 \ll V,\ \ddot{\phi} \ll \frac{dV}{d\phi}$$ slow roll approximation

$$\Rightarrow \begin{cases} \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_G^2} V(\phi) \\ 3\frac{\dot{a}}{a}\dot{\phi} = -\frac{dV}{d\phi} \end{cases}$$

$$\frac{\dot{a}}{a} = \frac{V(\phi)^{1/2}}{\sqrt{3}M_G} \qquad 3\dot{\phi}\frac{\dot{a}}{a} = \frac{\sqrt{3}V(\phi)^{1/2}}{M_G}\dot{\phi} = -\frac{dV}{d\phi}$$

$$V = m^2\phi^2/2 \quad \Rightarrow \quad \sqrt{3}\frac{1}{M_G}\frac{m}{\sqrt{2}}\phi\dot{\phi} = -m^2\phi$$

$$\dot{\phi} = -\frac{\sqrt{2}}{\sqrt{3}}M_G m$$

$$\phi - \phi_i = -\frac{\sqrt{2}}{\sqrt{3}}M_G m(t - t_i)$$

$$\frac{\dot{a}}{a} = \frac{1}{\sqrt{3}M_G}\frac{1}{\sqrt{2}}m\phi = \frac{m}{\sqrt{6}M_G}\left(\phi_i - \frac{\sqrt{2}}{\sqrt{3}}M_G m(t-t_i)\right)$$

$$\begin{aligned} \ln\frac{a}{a_i} &= \frac{m}{\sqrt{6}M_G}\left(\phi_i(t-t_i) - \frac{1}{\sqrt{6}}M_G m(t-t_i)^2\right) \\ &= \frac{1}{4M_G^2}(\phi_i^2 - \phi^2) \end{aligned}$$

$$a = a_i \exp\left[\frac{1}{4M_G^2}(\phi_i^2 - \phi^2)\right]$$

3.3 Slow Roll Condition

$$\dot{\phi} \simeq -\frac{\frac{dV}{d\phi}}{3\frac{\dot{a}}{a}} \simeq -\frac{V'}{3H} \qquad \ddot{\phi} \simeq -\frac{V''}{3H}\dot{\phi} \simeq \frac{V''V'}{9H^2}$$

(1) $|\ddot{\phi}| \ll \left|\frac{dV}{d\phi}\right|$

$$\Rightarrow \quad \left|\frac{V''V'}{9H^2}\right| \ll |V'| \qquad |V''| \ll 9H^2 = 3\frac{V}{M_G^2}$$

$$\boxed{\eta \equiv \frac{V''}{V}M_G^2} \qquad \Rightarrow \quad |\eta| \ll 1$$

(2) $\frac{1}{2}\dot{\phi}^2 \ll V$

$$\Rightarrow \quad \frac{(V')^2}{9H^2} \ll 2V \qquad (V')^2 \ll 18H^2V = 6\frac{V^2}{M_G^2}$$

$$\boxed{\epsilon \equiv \frac{1}{2}\left(\frac{V'}{V}\right)^2 M_G^2} \qquad \Rightarrow \quad \epsilon \ll 1$$

For chaotic inflation $V = \frac{1}{2}m^2\phi^2$

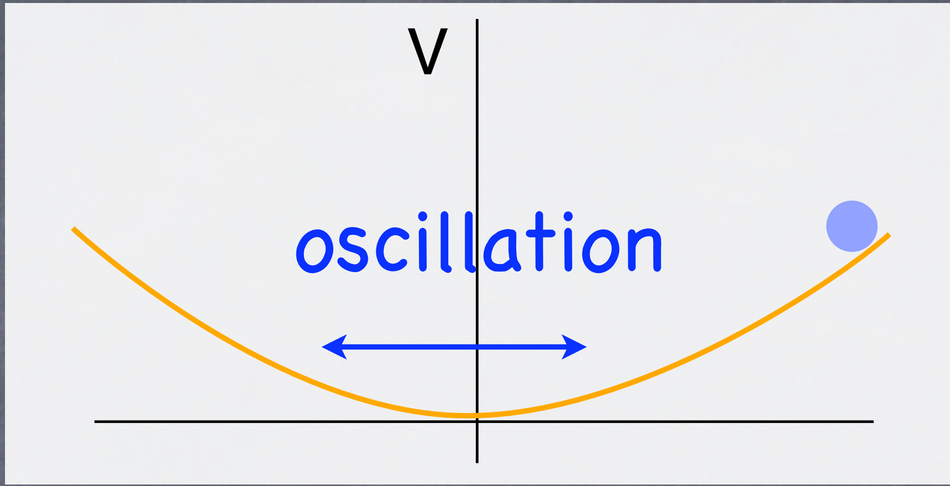
$$\eta = \frac{m^2}{m^2\phi^2/2}M_G^2 \ll 1 \quad \Rightarrow \boxed{\phi \gg \sqrt{2}M_G}$$

$$\epsilon = \frac{1}{2}\left(\frac{m^2\phi}{m^2\phi^2/2}\right)^2 M_G^2 \ll 1 \quad \Rightarrow \boxed{\phi \gg \sqrt{2}M_G}$$

Inflation ends when $\eta \simeq 1 \text{ or } \epsilon \simeq 1$

$$\Rightarrow \quad \boxed{\phi_f \simeq \sqrt{2}M_G}$$

3.4 After Inflation



$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -\frac{dV}{d\phi}$$

$$\times\dot{\phi} \quad \ddot{\phi}\dot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi}^2 = -\frac{dV}{d\phi}\dot{\phi} \quad \Rightarrow \quad \frac{d}{dt}\Bigg(\underbrace{\frac{1}{2}\dot{\phi}^2 + V}_{\rho_\phi}\Bigg) = -3\frac{\dot{a}}{a}\dot{\phi}^2$$

$V = \frac{1}{2}m^2\phi^2$ assume $H = \dot{a}/a \ll m$

We can neglect the cosmic expansion

$$\ddot{\phi} = -m^2\phi \quad \Rightarrow \quad \phi = A\sin(mt+\delta)$$

$$\frac{1}{2}\dot{\phi}^2 = \frac{1}{2}m^2A^2\cos^2(mt+\delta)$$

To see the effect of cosmic expansion, take average over one oscillation

$$\left\langle \frac{1}{2}\dot{\phi}^2 \right\rangle = \frac{1}{4}m^2 A^2 = \langle V(\phi) \rangle = \frac{1}{2}\rho_\phi$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V\right) = -3\frac{\dot{a}}{a}\dot{\phi}^2$$

$$\boxed{\frac{d}{dt}\rho_\phi = -3\frac{\dot{a}}{a}\rho_\phi \quad \Rightarrow \quad \rho_\phi \propto a^{-3}}$$

Inflaton oscillation behaves as matter

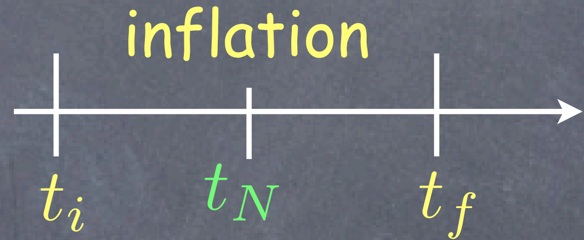
Oscillation $\Rightarrow$ Reheating

particle creation

3.5 N-efold

- The scale factor increases by e^N from t_N to t_f

$$\frac{dN}{dt_N} = \frac{d}{dt_N} \ln\left(\frac{a(t_f)}{a(t_N)}\right) = -\frac{\dot{a}}{a} = -H$$



$$\Rightarrow \ N = \int_{t_N}^{t_f} dt(-H) \simeq \int_{\phi_N}^{\phi_f} \left(-\frac{3H}{V'} d\phi\right) (-H)$$

$$d\phi/dt = -V'/3H$$

$$\boxed{\Rightarrow \ N \simeq \int_{\phi_N}^{\phi_f} \frac{V}{V' M_G^2} d\phi}$$

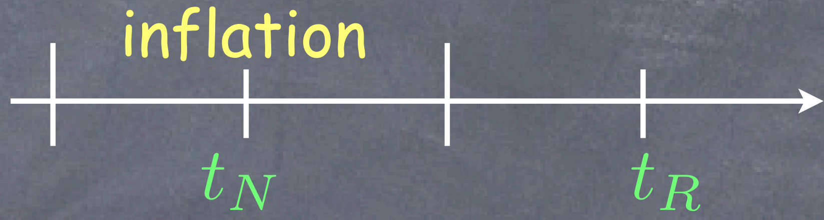
Chaotic inflation $V = \frac{1}{2}m^2\phi^2$

$$
\begin{aligned}
N &= -\int_{\phi_N}^{\phi_f} \frac{m^2\phi^2/2}{m^2\phi M_G^2} d\phi \\
&= \frac{-1}{2M_G^2} \int_{\phi_N}^{\phi_f} d\phi\phi \\
&= \frac{1}{4M_G^2}[\phi_N^2 - \phi_f^2] \\
&= \frac{1}{4M_G^2}[\phi_N^2 - 2M_G^2]
\end{aligned}
$$

$$N_{tot} = \int_{\phi_i}^{\phi_f} \frac{V}{V' M_G^2} d\phi = \frac{1}{4M_G^2}[\phi_i^2 - 2M_G^2] \sim \left(\frac{M_G}{m}\right)^2$$

$$\sim 10^{10}$$

3.5 Relation between N and Cosmological Scale L



Hubble radius at t_N corresponds to the present scale L

- At the end of inflation

$$H^{-1}(t_N) \;\rightarrow\; e^N H^{-1}(t_N)$$

- Between t_f and t_R (reheating epoch)
  Inflaton oscillation

$$\rho_\phi = \frac{1}{2}\Phi_f^2 m^2 \left(\frac{a(t)}{a(t_f)}\right)^{-3} \qquad \Phi \ \text{: amplitude of oscillation}$$

$$\rho_\phi(t_R) \simeq \frac{\pi^2}{30} g_* T_R^4$$

$$\Rightarrow \quad \left(\frac{a(t_R)}{a(t_f)}\right)^3 \simeq \frac{\frac{1}{2}\Phi_f^2 m^2}{\frac{\pi^2}{30} g_* T_R^4} = \frac{M_G^2 m^2}{\frac{\pi^2}{30} g_* T_R^4} \qquad \leftarrow \ \Phi_f \simeq \sqrt{2} M_G$$

At t=t$_R$ the entropy of the region corresponding to $H^{-1}(t_N)$

$$S_N = H^{-3}(t_N) e^{3N} \left(\frac{a(t_R)}{a(t_f)}\right)^3 s(T_R) \qquad s(t_R) = \frac{2\pi^2}{45} g_* T_R^3$$

At present the entropy inside the region with scale L

$$S_L = 10^{87} \left(\frac{L}{3000 h^{-1}\mathrm{Mpc}}\right)^3$$

$$\boxed{\Rightarrow \quad N = 54 + \ln\left(\frac{L}{3000 h^{-1}\mathrm{Mpc}}\right) + \frac{1}{3}\ln\left(\frac{T_R}{10^{10}\mathrm{GeV}}\right) + \frac{1}{3}\ln\left(\frac{m}{10^{13}\mathrm{GeV}}\right)}$$

4. Generation of Density Fluctuations

4.1 Fluctuations of scalar field during inflation

massless ϕ

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\triangle\phi = 0$$

$$ds^2 = dt^2 - e^{2Ht}d\vec{x}^2$$

$$\Rightarrow \quad \phi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\int d^3k[a_k^\dagger \psi_k^*(t)e^{-i\vec{k}\vec{x}} + a_k\psi_k(t)e^{i\vec{k}\vec{x}}]$$

$$[a_k, a_q^\dagger] = \delta^{(3)}(\vec{k}-\vec{q})$$

c.f. Minkowski space

$$\phi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2k_0}}[a_k^\dagger e^{+ik_0t-i\vec{k}\vec{x}} + a_k e^{-ik_0t+i\vec{k}\vec{x}}]$$

$$\psi_k \quad \rightarrow \quad \frac{e^{-ik_0t}}{\sqrt{2k_0}}$$

Remark:

k : comoving momentum

$p = e^{-Ht}k$: physical momentum

$$\boxed{\ddot{\psi}_k + 3H\dot{\psi}_k + k^2 e^{-2Ht}\psi_k = 0}$$

$$\eta = -H^{-1}e^{-Ht}, \quad \psi = \eta^{3/2}u$$

$$\Rightarrow \; u'' + \frac{1}{\eta}u' + \left(k^2 - \frac{9}{4\eta^2}\right)u = 0$$

Bessel differential eq.

General sol.

$$\psi_k(t) = \frac{\sqrt{\pi}}{2}H\eta^{3/2}[C_1(k)H^{(1)}_{3/2}(k\eta) + C_2(k)H^{(2)}_{3/2}(k\eta)]$$

$H^{(1,2)}_{3/2}$: Hankel function

$$H^{(2)}_{3/2} = (H^{(1)}_{3/2})^* = \sqrt{\frac{2}{\pi x}}e^{-ix}\left(1 + \frac{1}{ix}\right)$$

$C_1(k), C_2(k)$?

Quantization in de Sitter space should be the same as that in Minkowski space in the limit of $k \to \infty$

$$\psi_k \quad \to \quad \frac{-1}{\sqrt{2k}}\left[C_1(k)e^{-iH^{-1}k+ikt} + C_2(k)e^{iH^{-1}k-ikt}\right]$$

$$= \quad \frac{e^{-ikt}}{\sqrt{2k}}$$

$$\Rightarrow \quad C_1(k) \to 0 \quad C_2(k) \to 1$$

$$\Rightarrow \boxed{\psi_k(t) = \frac{\sqrt{\pi}}{2} H\eta^{3/2} H^{(2)}_{3/2}}$$

For cosmologically interesting fluctuations

$k \gg H$ at the start of the inflation

$$\Rightarrow \boxed{C_2 = 1, \quad C_1 = 0}$$

$$\Rightarrow \psi_k(t) = \frac{-iH}{k\sqrt{2k}} \left(1 + \frac{k}{iH} e^{-Ht}\right) \exp\left(-\frac{ik}{H} e^{-Ht}\right)$$

Large t

$$\psi_k(t) \to \frac{-iH}{k\sqrt{2k}}$$

no oscillation

$$\langle \phi^2 \rangle = \int |\psi_k|^2 d^3k = \frac{1}{(2\pi)^3} \int \left(\frac{e^{-2Ht}}{2k} + \frac{H^2}{2k^3}\right) d^3k$$

$$\langle \phi^2 \rangle = \frac{1}{(2\pi)^3} \int \left(\frac{1}{2} + \frac{H^2}{2p^2}\right) \frac{d^3p}{p} \qquad p = ke^{-Ht}$$

vacuum fluctuations

$$\langle \phi^2 \rangle = \frac{H^2}{(2\pi)^2} \int d \ln \left(\frac{p}{H} \right)$$

$$\Rightarrow \boxed{\delta\phi \simeq \frac{H}{2\pi}}$$

Fourier mode $\qquad \delta\phi_k = \dfrac{H}{\sqrt{2} k^{3/2}}$

Density Fluctuations

Inflaton fluctuations $\Rightarrow$ metric fluctuations

$\Rightarrow$ density fluctuations

$$\boxed{ds^2 = (1+2\Psi)dt^2 - a^2(1+2\Phi)dx^2}$$ newtonian gauge

$$\Rightarrow \boxed{\begin{aligned} \Psi &= -\Phi \\ \Phi &= H_I \frac{\delta\phi}{\dot{\phi}}\left(1 - \frac{\dot{a}}{a^2}\int_o^t a\,dt\right) \end{aligned}} \qquad \Phi = \begin{cases} \frac{2}{3}H\frac{\delta\phi}{\dot{\phi}} & \text{(RD)} \\ \frac{3}{5}H\frac{\delta\phi}{\dot{\phi}} & \text{(MD)} \end{cases}$$

Poisson Equation

$$\triangle\Phi = -4\pi G a^2 \delta\rho \longrightarrow k^2\Phi_k = 4\pi G a^2 \delta\rho_k$$

$$\Rightarrow \boxed{\frac{\delta\rho_k}{\rho} = \frac{2}{3}\frac{k^2}{a^2H^2}\Phi_k = \frac{2}{5}\frac{k^2}{a^2H^2}H_{\rm I}\frac{\delta\phi_k}{\dot{\phi}}}$$

$$\boxed{\frac{\delta\rho_k}{\rho} = \frac{2}{3}\frac{k^2}{a^2H^2}\Phi_k = \frac{2}{5}\frac{k^2}{a^2H^2}H_{\mathrm{I}}\frac{\delta\phi_k}{\dot{\phi}}}$$

$$\delta\phi_k = \frac{H_{\mathrm{I}}}{\sqrt{2}k^{3/2}}$$

Power spectrum

$$\boxed{P(k) = \left(\frac{\delta\rho_k}{\rho}\right)^2 = \frac{2}{25}\left(\frac{k^2}{a^2H^2}\right)^2\frac{H_{\mathrm{I}}^4}{(\dot{\phi})^2}k^{-3} \propto k}$$

Harrison-Zeldovich
scale invariant spectrum

at Horizon crossing $k/a = H$

$$\left(\frac{k^3}{2\pi^2}P(k)\right)^{1/2} \simeq \frac{1}{5\pi}\frac{3H_{\mathrm{I}}^3}{V'} = \frac{V^{3/2}}{5\sqrt{3}\pi V' M_G^3} \longleftarrow \dot{\phi} \simeq -\frac{V'}{3H_{\mathrm{I}}}$$

$\sim 10^{-5} \;\rightarrow\; m_\phi \sim 10^{13}\mathrm{GeV}$ for chaotic inf.

Spectral Index $\quad P(k) \propto k^{n_s}$

$$n_s = 1 + 2\frac{d\ln(V^{3/2}/V')}{d\ln k}$$

$$d\ln k = -dN = -\frac{V}{V' M_G^2} d\phi$$

$$N \simeq 60 + \ln(k^{-1}/3000h^{-1}\mathrm{Mpc})$$

$$N \simeq \int d\phi (V/V') M_G^{-2}$$

$$\begin{aligned} n_s &= 1 - 2\frac{M_G^2 d\ln(V^{3/2}/V')}{(V/V')d\phi} = 1 - 2M_G^2 \frac{V'}{V}\left(\frac{3V'}{V} - \frac{V''}{V'}\right) \\ &= 1 - 6\varepsilon + 2\eta \end{aligned}$$

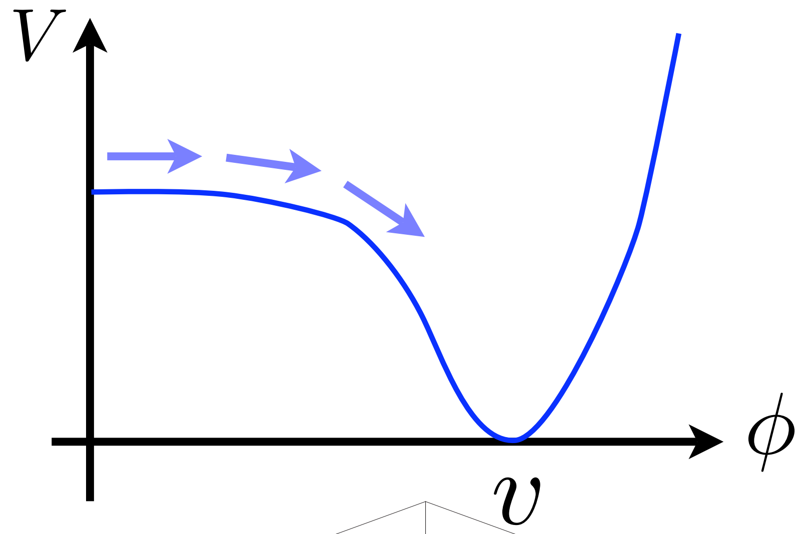
$$\boxed{n_s = 1 - 6\varepsilon + 2\eta}$$

$n_s \simeq 0.96$ for chaotic inf.

Other Inflation Models

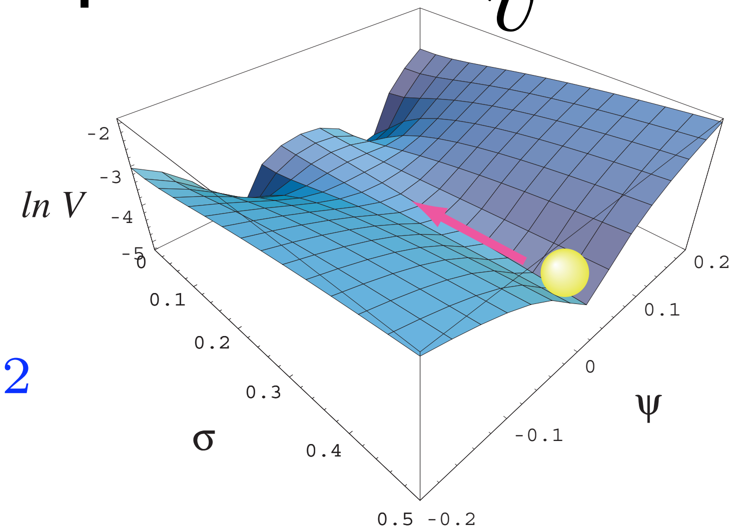
New Inflation Model

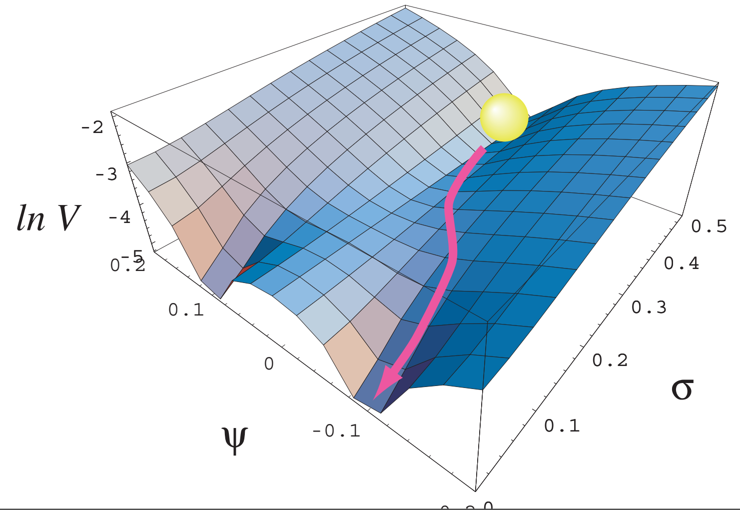
e.g. $V = \mu^4 \left(1 - \left(\frac{\phi}{v}\right)^n\right)^2 \qquad n \geq 4$



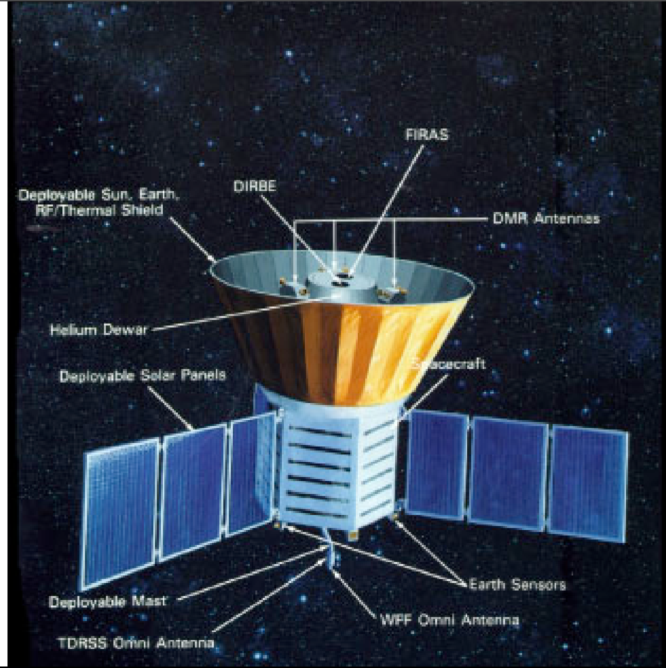
Hybrid Inflation Model

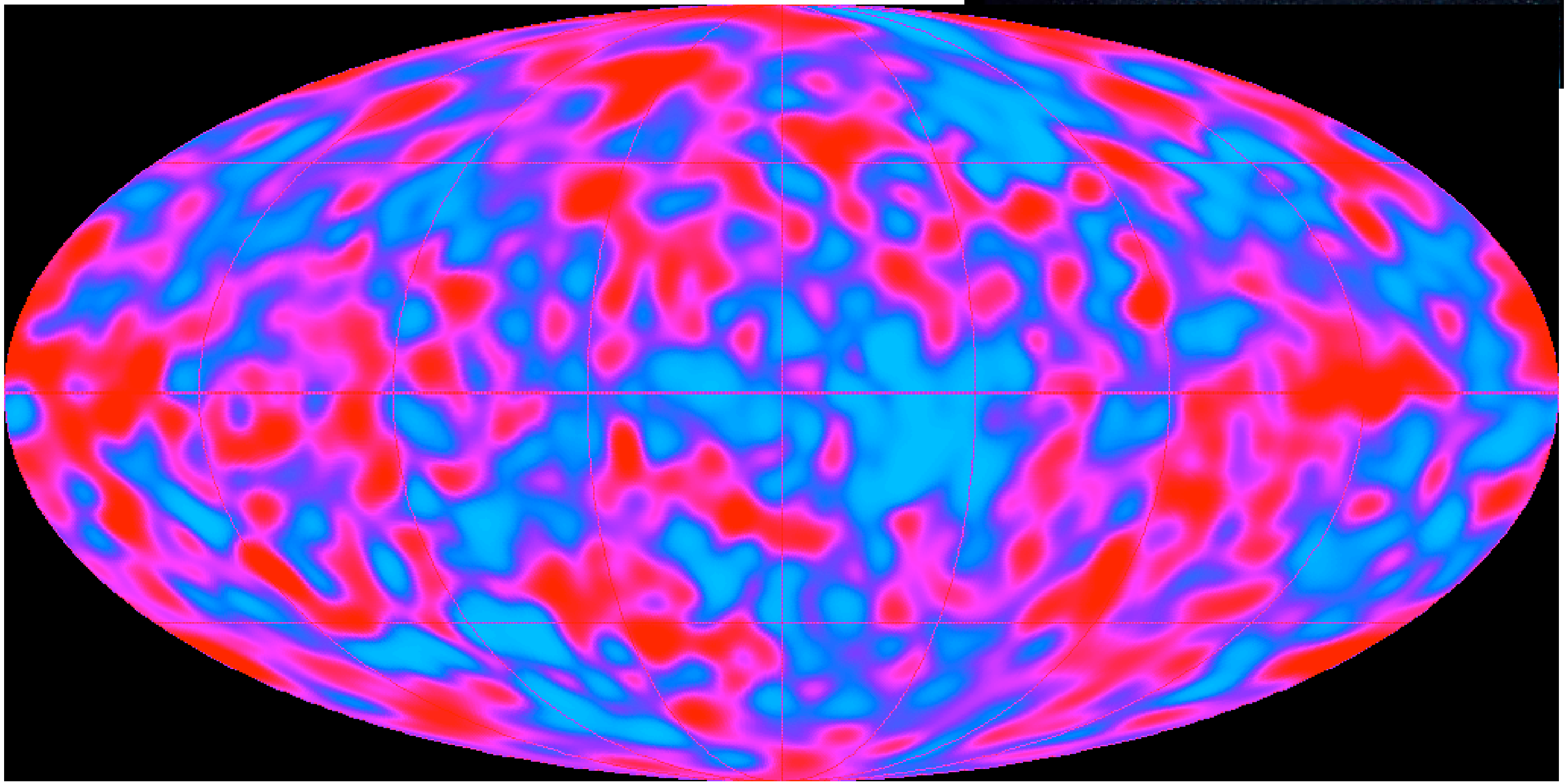
$$V = (\mu^2 - \lambda\psi^2)^2 + \kappa\psi^2\sigma^2 + \frac{1}{2}m_\sigma^2\sigma^2$$





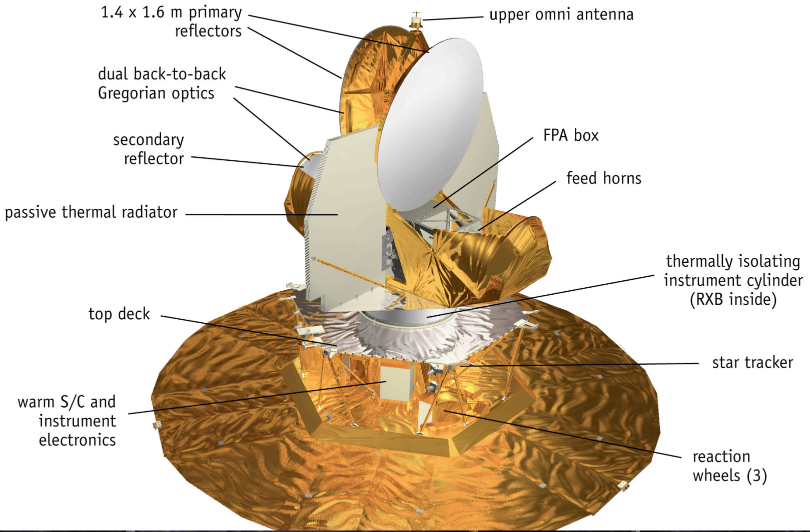
COBE

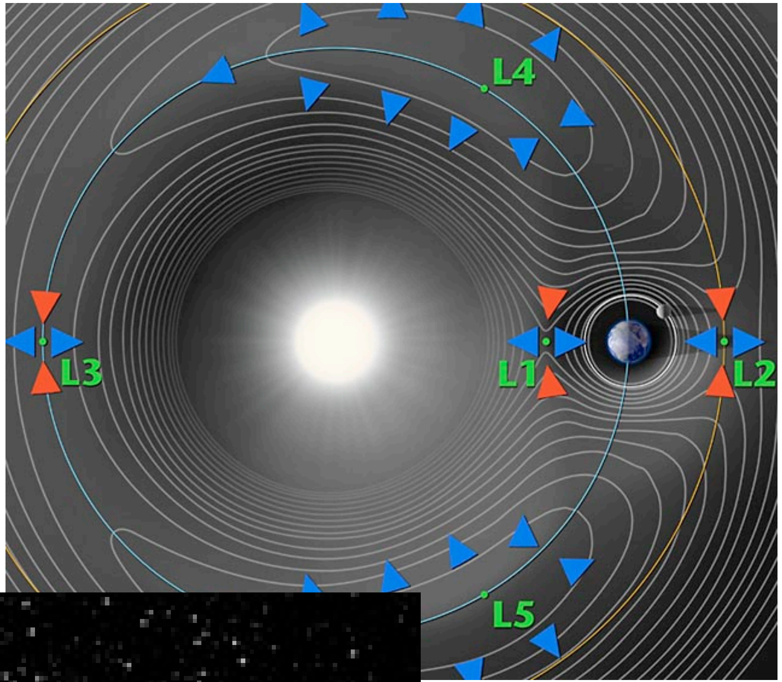


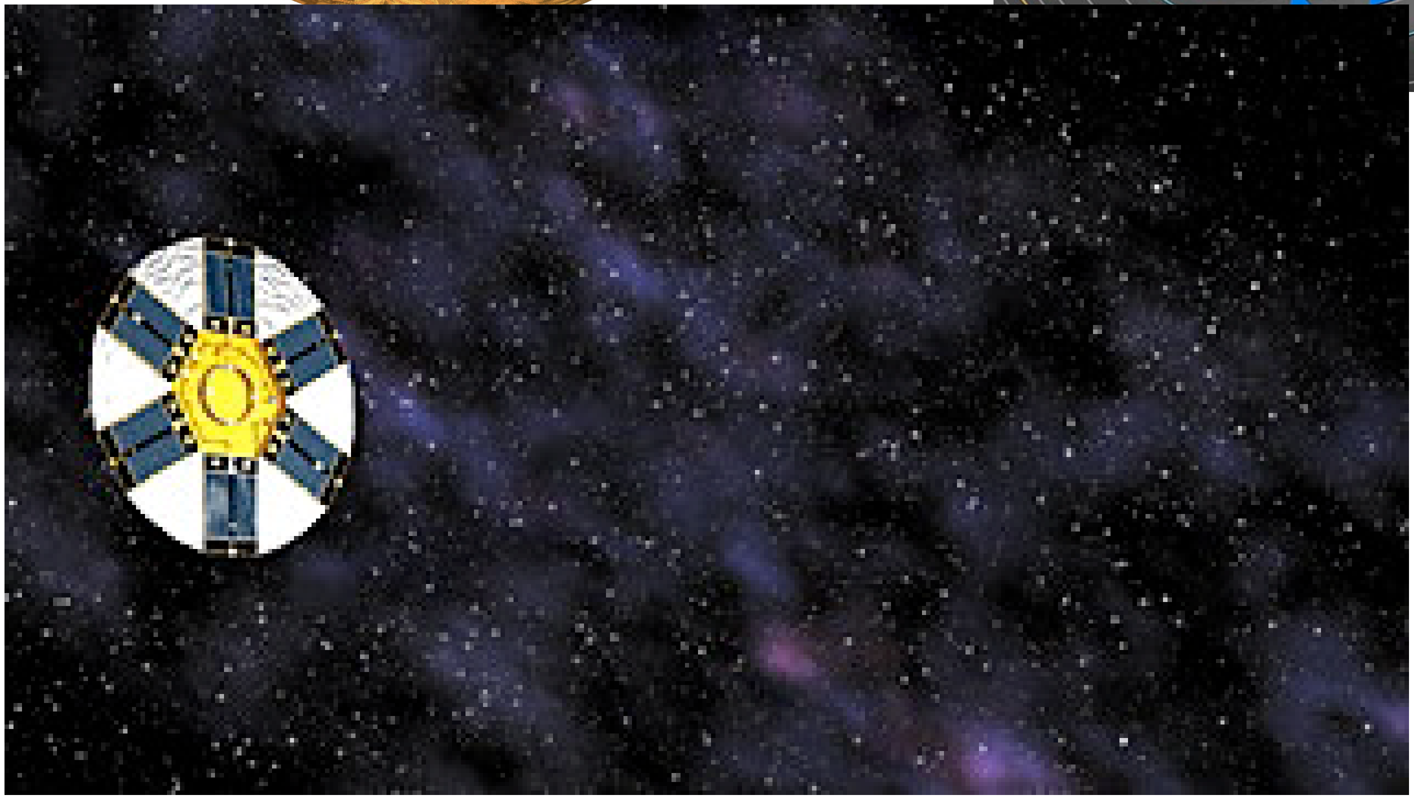


http://map.gsfc.nasa.gov/product/cobe

WMAP (Wilkinson Microwave Anisotropy Probe)







http://map.gsfc.nasa.gov/

WMAP Full Sky Map