Cosmology

Masahiro Kawasaki

- Standard Cosmology
- Inflationary Universe
- Gravitino Problem

unit $\hbar = c = k_B = 1$

Standard Cosmology

1. Standard Model

- Cosmological Principle

  - The universe is spatially homogeneous.
  - The universe is isotropic.

⇒ Robertson-Walker Metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

$a(t)$ scale factor

$$\begin{cases} K > 0 & \text{closed universe} \\ K = 0 & \text{flat universe} \\ K < 0 & \text{open universe} \end{cases}$$

Cosmological Principle

Energy-Momentum Tensor

= Perfect Fluid Form

$$\boxed{T^{\mu\nu} = -pg^{\mu\nu} + (\rho + p)u^\mu u^\nu}$$

$$u^0 = 1, \; u^i = 0$$

Einstein Equation

$$\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}}$$

Λ Cosmological Constant

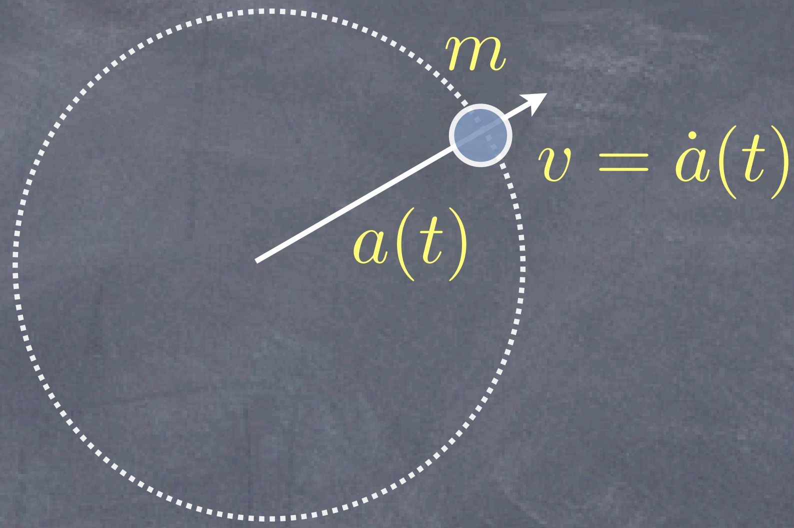
$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a + \frac{\Lambda}{3}a$$

$$\frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3)$$

two independent equations

Newtonian Picture



Energy Conservation

$$\frac{1}{2}mv^2 - \frac{GMm}{a} = mE = \text{const}$$

$$M = \frac{4}{3}\pi a^3 \rho \quad \text{total mass}$$

$$\Rightarrow \quad \frac{\dot{a}^2}{2} - \frac{4\pi G\rho}{3}a^2 = E \equiv -\frac{K}{2}$$

$$\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{8\pi}{3}G\rho$$

1st Law in Thermodynamics

$$dE = -pdV$$

$$E = \rho a^3 \quad V = a^3$$

$$\Rightarrow \quad \frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3)$$

2. Cosmological Parameters

Friedmann Equation $$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

- Hubble Parameter $$\boxed{H \equiv \left(\frac{\dot{a}}{a}\right)}$$

  The present Hubble parameter = Hubble constant H_0

  obs: $H_0 = h_0 \times (100\mathrm{km/s/Mpc})$ $\mathrm{Mpc} \simeq 3 \times 10^{24}\mathrm{cm}$

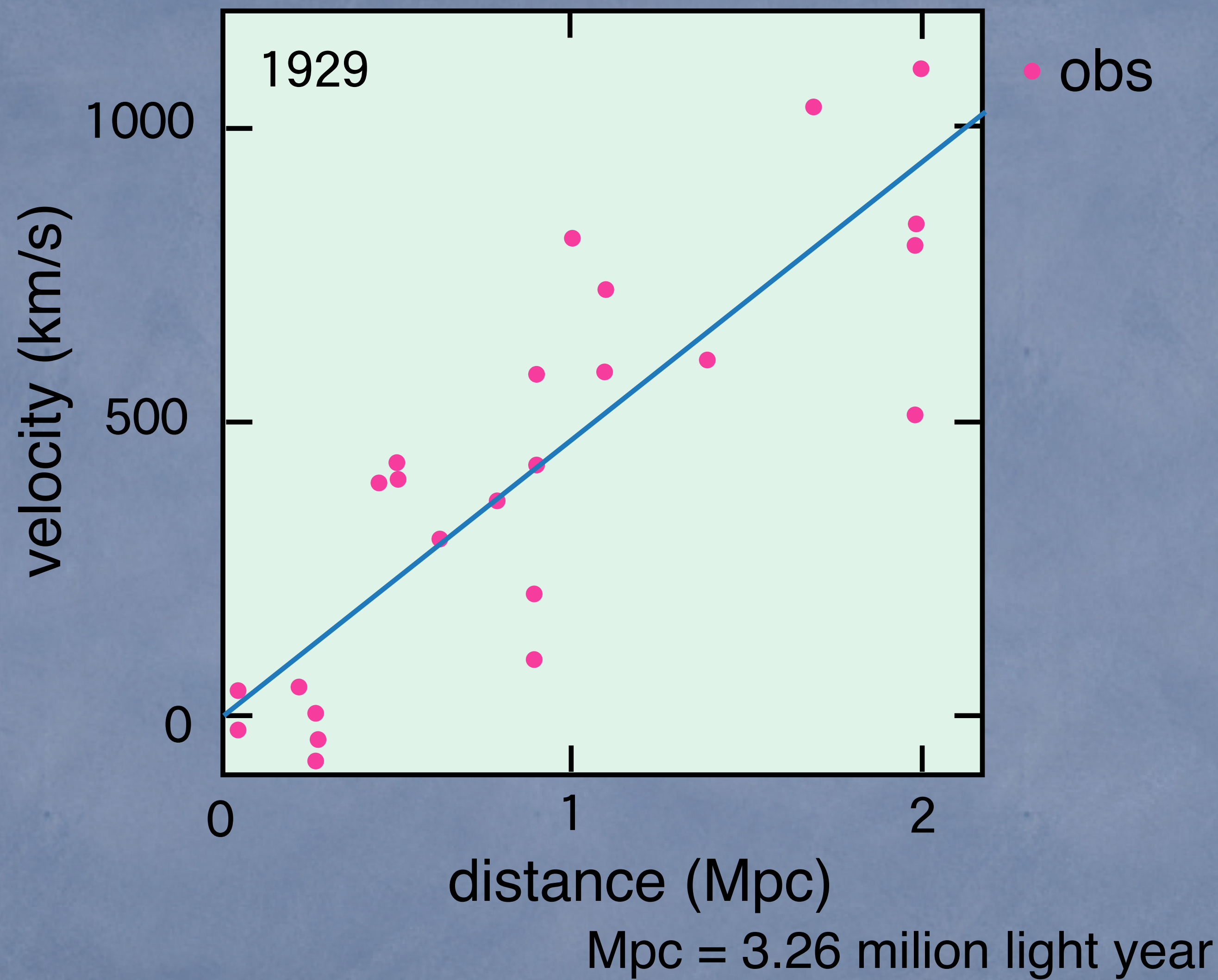
  $h_0 \simeq 0.7 \pm 0.1$

- Density Parameter $$\boxed{\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2}}$$

  critical density $$\rho_c \equiv \frac{3H^2}{8\pi G}$$

  $\rho_{c,o} = 1.053 \times 10^{-5} h^2 \ \mathrm{GeV cm}^{-3}$

- Cosmological constant $$\boxed{\lambda = \frac{\Lambda}{3H^2}}$$

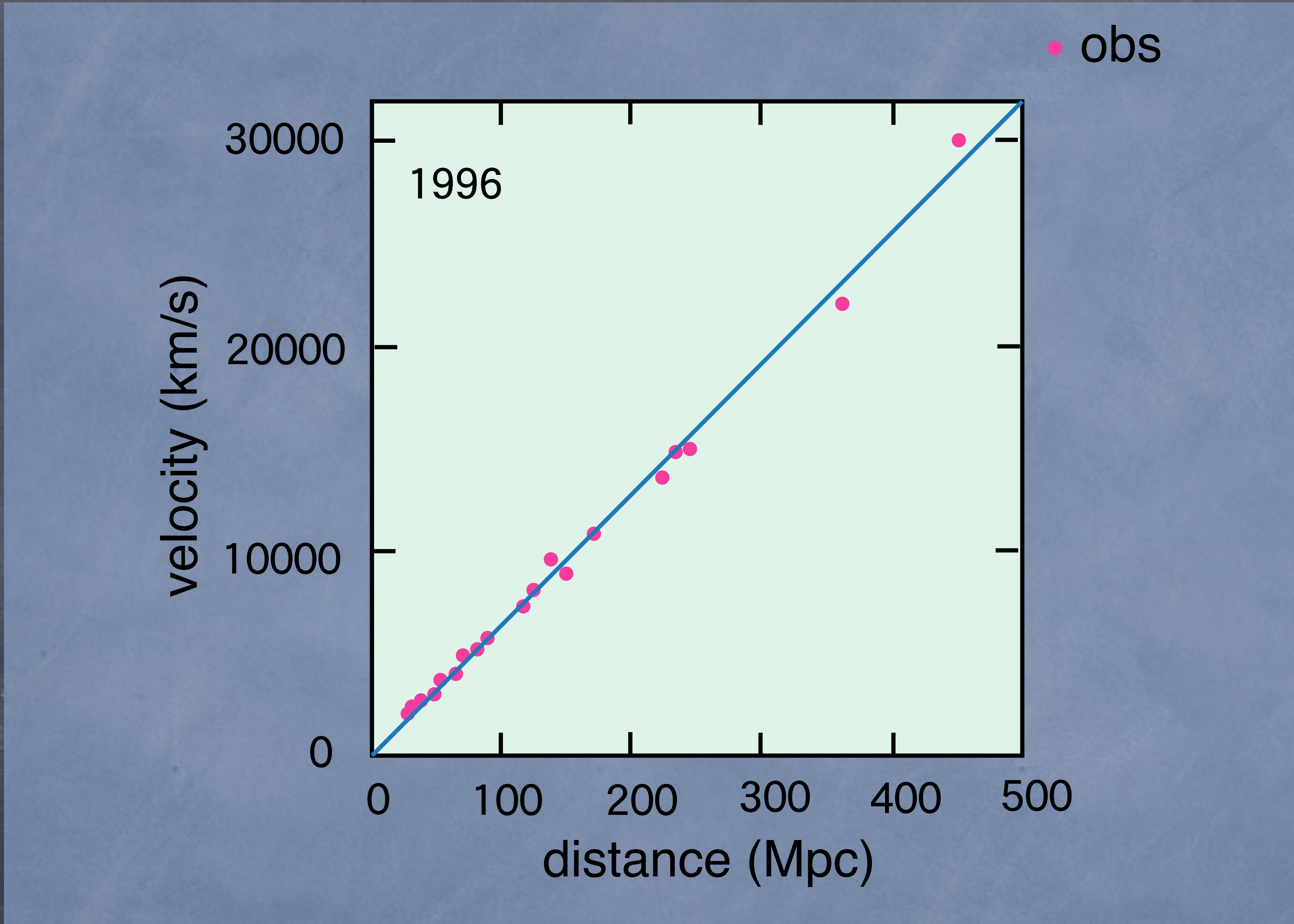


(velocity) = H_0 X (distance)

H_0 Hubble Constant = 465km/s/Mpc

Hubble Space Telescope

http://hubble.nasa.gov/



(velocity) = H_0 X (distance)

H_0 Hubble Constant = 68-75km/s/Mpc

Friedmann Equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho \quad \Rightarrow \quad H^2 + \frac{K}{a^2} - \lambda H^2 = \Omega H^2$$

$$\Rightarrow \quad \boxed{(\Omega + \lambda - 1)H^2 = \frac{K}{a^2}}$$

$$\boxed{\Omega + \lambda \begin{cases} > 1 \\ = 1 \\ < 1 \end{cases} \iff \begin{cases} K > 0 & \text{closed universe} \\ K = 0 & \text{flat universe} \\ K < 0 & \text{open universe} \end{cases}}$$

3. Density of the Universe

Einstein equation $$\boxed{\frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3)}$$

equation of state $$\boxed{p = w\rho}$$

$$\Rightarrow \quad \frac{d}{dt}(a^3\rho) = a^3\frac{d\rho}{dt} + \rho\frac{d}{dt}(a^3) = -w\rho\frac{d}{dt}(a^3)$$

$$\frac{d\rho}{\rho} = -(1+w)\frac{1}{a^3}d(a^3)$$

$$\Rightarrow \quad \boxed{\rho \propto a^{-3(1+w)}}$$

In cosmology, three kinds of densities are important

- ρ_m : Matter (non-relativistic, non-thermal)

$$\rho_m \propto a^{-3} \qquad w = 0$$

- ρ_R : Radiation (relativistic particles)

$$\rho_R \propto a^{-4} \qquad w = 1/3$$

- ρ_{DM} : Dark Energy $\quad w < 0$

$w = -1 \ \Rightarrow$ Cosmological constant $\ \rho_{DM} \propto a^0$

$$\Rightarrow \qquad \rho = \rho_m + \rho_R + \rho_{DM}$$

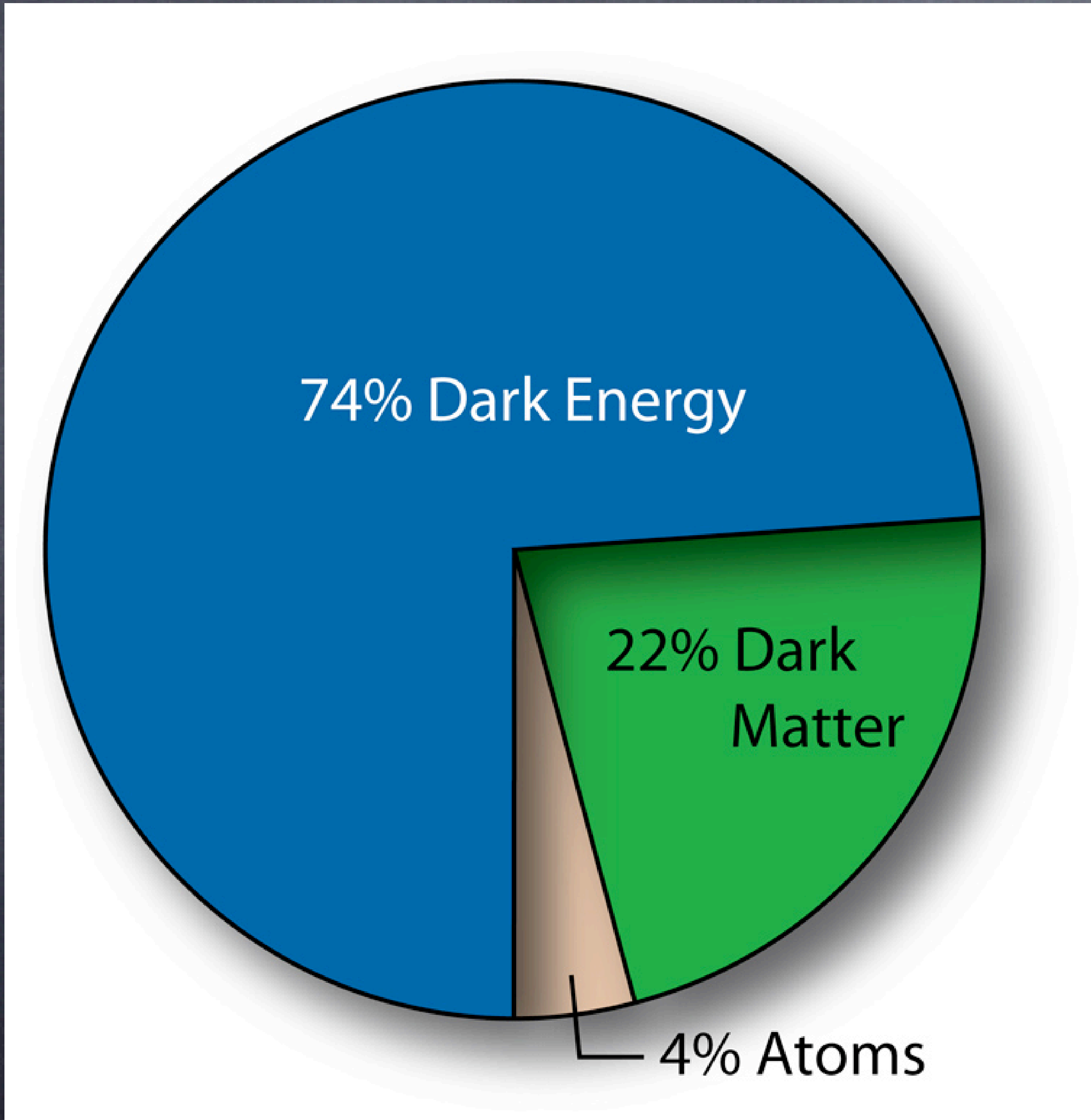
The present Universe

particle	temp (K)	density (1/cm3)	density (eV/cm3)	Ωh^2
photon	2.73	415	0.23	2.2×10^{-5}
neutrino	1.95	113X3	0.052X3	$4.9 \times 10^{-6} \times 3$
baryon	-	2.5×10^{-7}	250	0.02

The present Universe

particle	temp (K)	density (1/cm3)	density (eV/cm3)	Ωh^2
photon	2.73	415	0.23	2.2×10^{-5}
neutrino	1.95	113X3	0.052X3	$4.9\times10^{-6}\times3$
baryon	-	2.5×10^{-7}	250	0.02
dark matter	-	?		0.105
dark energy	-	?		0.38

Resent WMAP result

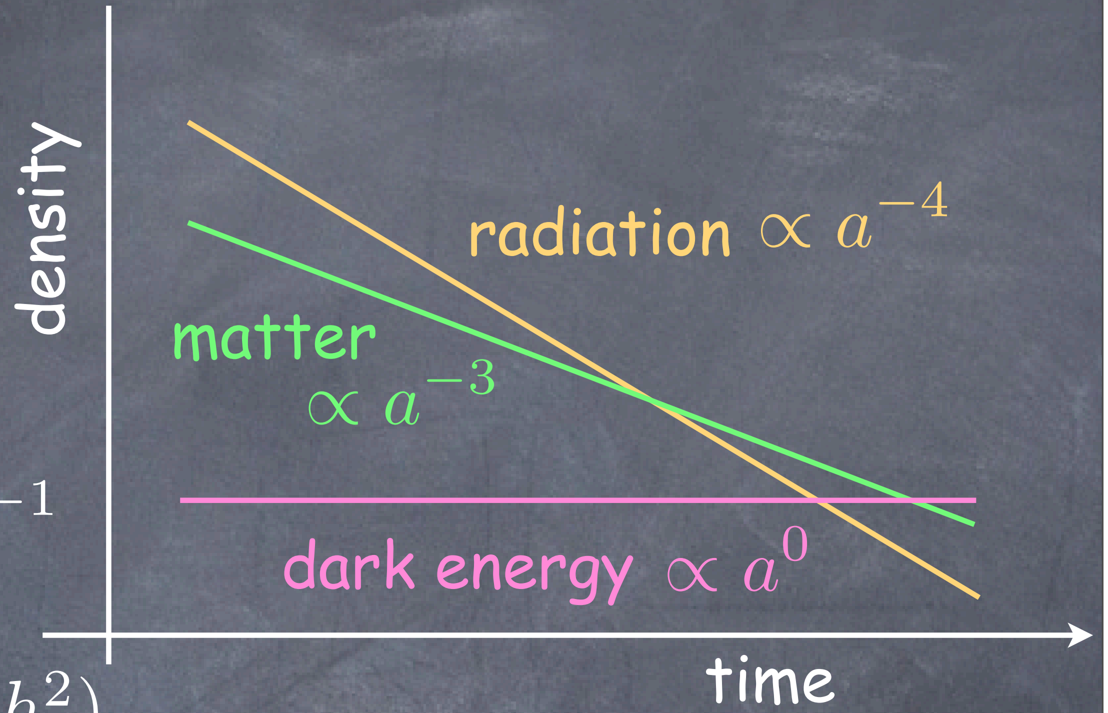


Matter vs Radiation

$$\frac{\rho_R}{\rho_m} = \frac{\Omega_{\gamma,0}+\Omega_{\nu,0}}{\Omega_B+\Omega_{DM}}\left(\frac{a}{a_0}\right)^{-1}$$

$$= 3.7\times 10^{-5}(\Omega_m h^2)^{-1}\left(\frac{a}{a_0}\right)^{-1}$$

$$= 1 \quad \Rightarrow \quad \frac{a_0}{a} = 2.7\times 10^4(\Omega_m h^2)$$



$$(\Omega_m = \Omega_{B,0} + \Omega_{DM,0})$$

The early universe is radiation-dominated

$$\frac{a_0}{a} > \frac{a_0}{a_{eq}} = 2.7\times 10^4(\Omega_m h^2)$$

4. Matter Dominated Universe

Friedmann Equation $$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

$$\begin{cases} K = a_0^2(\Omega_m + \lambda - 1) \\ \Lambda = 3H_0^2\lambda \\ \frac{8\pi}{3}G\rho = \frac{8\pi}{3}G\rho_0\left(\frac{a}{a_0}\right)^{-3} = \Omega_m H_0^2\left(\frac{a}{a_0}\right)^{-3} \end{cases}$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = -H_0^2(\Omega_m + \lambda - 1)\left(\frac{a_0}{a}\right)^2 + H_0^2\Omega_m\left(\frac{a_0}{a}\right)^3 + H_0^2\lambda}$$

solved analytically for λ=0

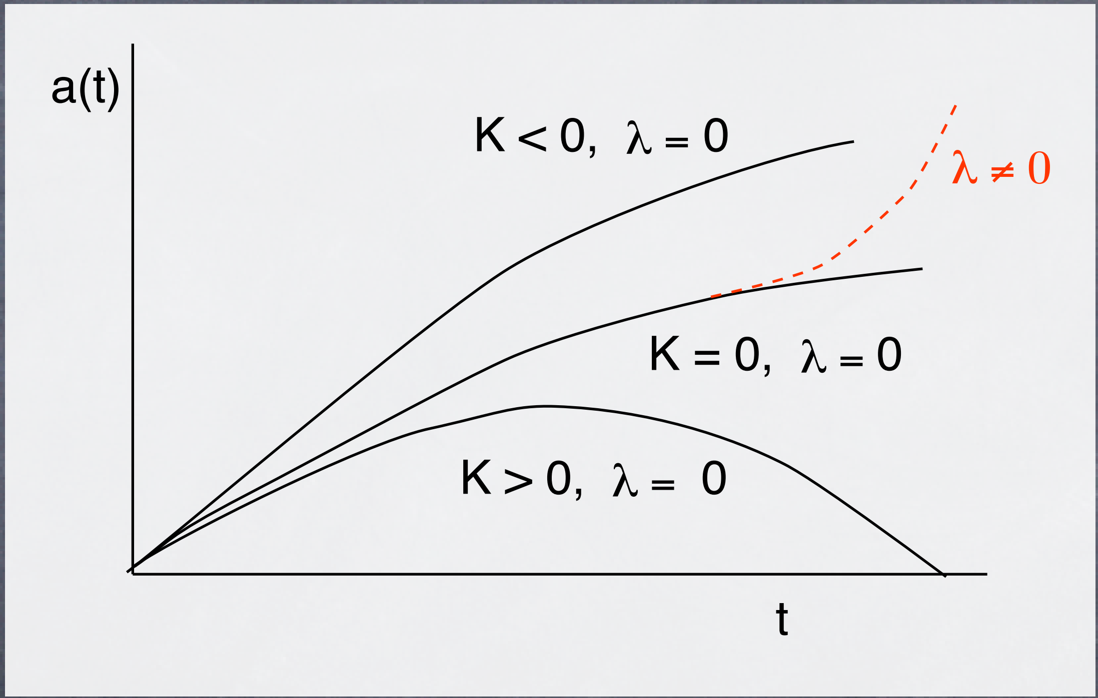
- K < 0
$$\begin{cases} \frac{a}{a_0} = \frac{1}{2}(1-\Omega_m)^{-1}\Omega_m(\cosh\eta - 1) \\ t = \frac{1}{2}H_0^{-1}\Omega_m(1-\Omega_m)^{-3/2}(\sinh\eta - \eta) \end{cases}$$

- K > 0
$$\begin{cases} \frac{a}{a_0} = \frac{1}{2}(\Omega_m - 1)^{-1}\Omega_m(1-\cos\eta) \\ t = \frac{1}{2}H_0^{-1}\Omega_m(\Omega_m - 1)^{-3/2}(\eta - \sin\eta) \end{cases}$$

- K = 0
$$\frac{a}{a_0} = \left(\frac{3H_0 t}{2}\right)^{2/3}$$

for small t

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\Omega_m\left(\frac{a_0}{a}\right)^3 \quad\Rightarrow\quad \frac{a}{a_0} = \left(\frac{3H_0\Omega_m^{1/2}t}{2}\right)^{2/3}$$



Λ dominated universe

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\lambda \quad\Rightarrow\quad a = a_0\exp(H_0\lambda^{1/2}(t-t_0))$$

5. Entropy of the Universe

Thermodynamics $$dS(V,T) = \frac{1}{T}d(\rho V) + \frac{p}{T}dV$$

$\rho(T),\ p(T)$ function of T

$$\boxed{S(V,T) = \frac{V}{T}(\rho(T) + p(T))}$$

Entropy of the Universe

$$\boxed{S = \frac{a^3}{T}(\rho(T) + p(T))}$$

Einstein eq. $$\frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3)$$

$$dS = \frac{1}{T}d(\rho a^3) + \frac{p}{T}d(a^3) \Rightarrow \frac{dS}{dt} = 0 \ \Rightarrow \ S \text{ const}$$

5. Entropy of the Universe

Thermodynamics $$dS(V,T) = \frac{1}{T}d(\rho V) + \frac{p}{T}dV$$

$\rho(T),\ p(T)$ function of T

$$\Rightarrow \quad \boxed{S(V,T) = \frac{V}{T}(\rho(T) + p(T))}$$

Entropy of the Universe

$$\boxed{S = \frac{a^3}{T}(\rho(T) + p(T))}$$

The universe expands adiabatically

$\Uparrow$

Einstein eq. $$\frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3)$$

$$dS = \frac{1}{T}d(\rho a^3) + \frac{p}{T}d(a^3) \quad \Rightarrow \quad \frac{dS}{dt} = 0 \ \Rightarrow \ S \text{ const}$$

Entropy of the Universe

$$\boxed{S = \frac{a^3}{T}(\rho(T) + p(T))}$$

Relativistic particle in thermal equilibrium

$$\rho = \frac{\pi^2}{30} g T^4 \qquad p = \frac{1}{3}\rho$$

entropy density

$$s = \frac{\pi^2}{30} g \left(1 + \frac{1}{3}\right) T^3 = \frac{2\pi^2}{45} g T^3$$

$$S = a^3 s = \text{const} \;\Rightarrow\; T^3 a^3 = \text{const}$$

$$\boxed{T \propto a^{-1}}$$

6. Redshift

Wavelength of light becomes longer as the universe expands

Geodesics of light

$$ds^2 = 0 = dt^2 - a^2(t)\frac{dr^2}{1-Kr^2}$$

- Light emitted at $t=t_1$ $r=r_1$ reaches $r=0$ at $t=t_0$

$$\int_{t_1}^{t_0}\frac{dt}{a(t)} = \int_0^{r_1}\frac{dr}{\sqrt{1-Kr^2}}$$

- Light emitted at $t=t_1+\delta t_1$ $r=r_1$ reaches $r=0$ at $t=t_0+\delta t_0$

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0}\frac{dt}{a(t)} = \int_0^{r_1}\frac{dr}{\sqrt{1-Kr^2}}$$

$$\Rightarrow \int_{t_1+\delta t_1}^{t_0+\delta t_0}\frac{dt}{a(t)} = \int_{t_1}^{t_0}\frac{dt}{a(t)} \qquad \Rightarrow \frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}$$

$$\Rightarrow \quad \frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}$$

Redshift

$$z \equiv \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{\lambda_0}{\lambda_1} - 1 = \frac{\delta t_0}{\delta t_1} - 1 = \frac{a(t_0)}{a(t_1)} - 1$$

$$\boxed{\frac{a(t)}{a(t_0)} = \frac{1}{1+z}}$$

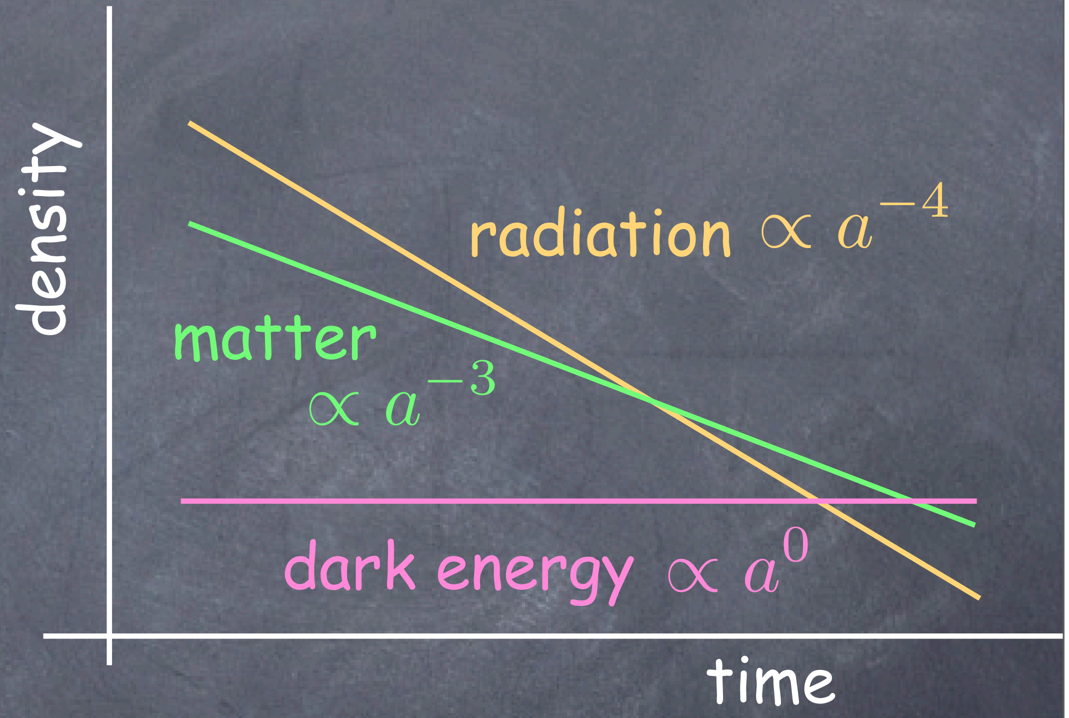
Momentum of photon (relativistic particle) decreases as 1/a

$$p_0 = \frac{a(t)}{a_0} p$$

7. Radiation Dominated Universe

The early universe is dominated by radiation

$$\boxed{\frac{a_0}{a} > \frac{a_0}{a_{eq}} = 2.7 \times 10^4 (\Omega_m h^2)}$$



Friedmann Equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \underbrace{\frac{K}{a^2}}_{\propto a^{-2}} - \underbrace{\frac{\Lambda}{3}}_{\propto a^{0}} = \underbrace{\frac{8\pi G}{3}\rho}_{\propto a^{-3 \text{ or } -4}}$$

We can neglect K- and Λ– terms

$$\Rightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \;\Rightarrow\; \left(\frac{\dot{T}}{T}\right)^2 = \frac{8\pi G}{3}\rho$$

$$T \propto 1/a$$

Total energy density

$$\rho = \frac{\pi^2}{30} g_* T^4$$

$$g_* = \sum_{\text{boson}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{fermion}} g_i \left(\frac{T_i}{T}\right)^4$$

for example

T=1MeV (γ, e, 3ν)

$$g_* = 2 + \frac{7}{8} \times 2 \times 2 + \frac{7}{8} \times 3 \times 2 = \frac{43}{4}$$

$$\gamma \qquad e^{\pm} \qquad 3\nu\bar{\nu}$$

$$\Rightarrow \left(\frac{\dot{T}}{T}\right)^2 = \frac{8\pi G}{3}\frac{\pi^2}{30} g_* T^4 = \frac{g_*}{3M_G^2}\frac{\pi^2}{30} T^4$$

$$M_G \equiv \frac{1}{\sqrt{8\pi G}} \simeq 2.2 \times 10^{18}\ \mathrm{GeV}$$

$$\boxed{\begin{aligned} t &= \left(\frac{45}{2\pi^2 g_*}\right)^{1/2} \frac{M_G}{T^2} \\ &\simeq 2.3\,\mathrm{sec}\, g_*^{-1/2} \left(\frac{T}{10^{10}\mathrm{K}}\right)^{-2} \\ &\simeq 1.7\,\mathrm{sec}\, g_*^{-1/2} \left(\frac{T}{\mathrm{MeV}}\right)^{-2} \end{aligned}}$$

$$\Rightarrow \ a(t) \propto t^{1/2}$$

8. Horizon

- Particle Horizon

  maximum travel distance of light emitted at t=0 until t

Geodesics of light $$ds^2 = 0 = dt^2 - a^2(t)\frac{dr^2}{1-Kr^2}$$

$$\boxed{\ell_H(t) \equiv a(t)\int_0^t \frac{dt'}{a(t')}}$$

$$a(t) \propto t^m \quad \begin{cases} m = 1/2 & (RD) \\ \quad = 2/3 & (MD) \end{cases}$$

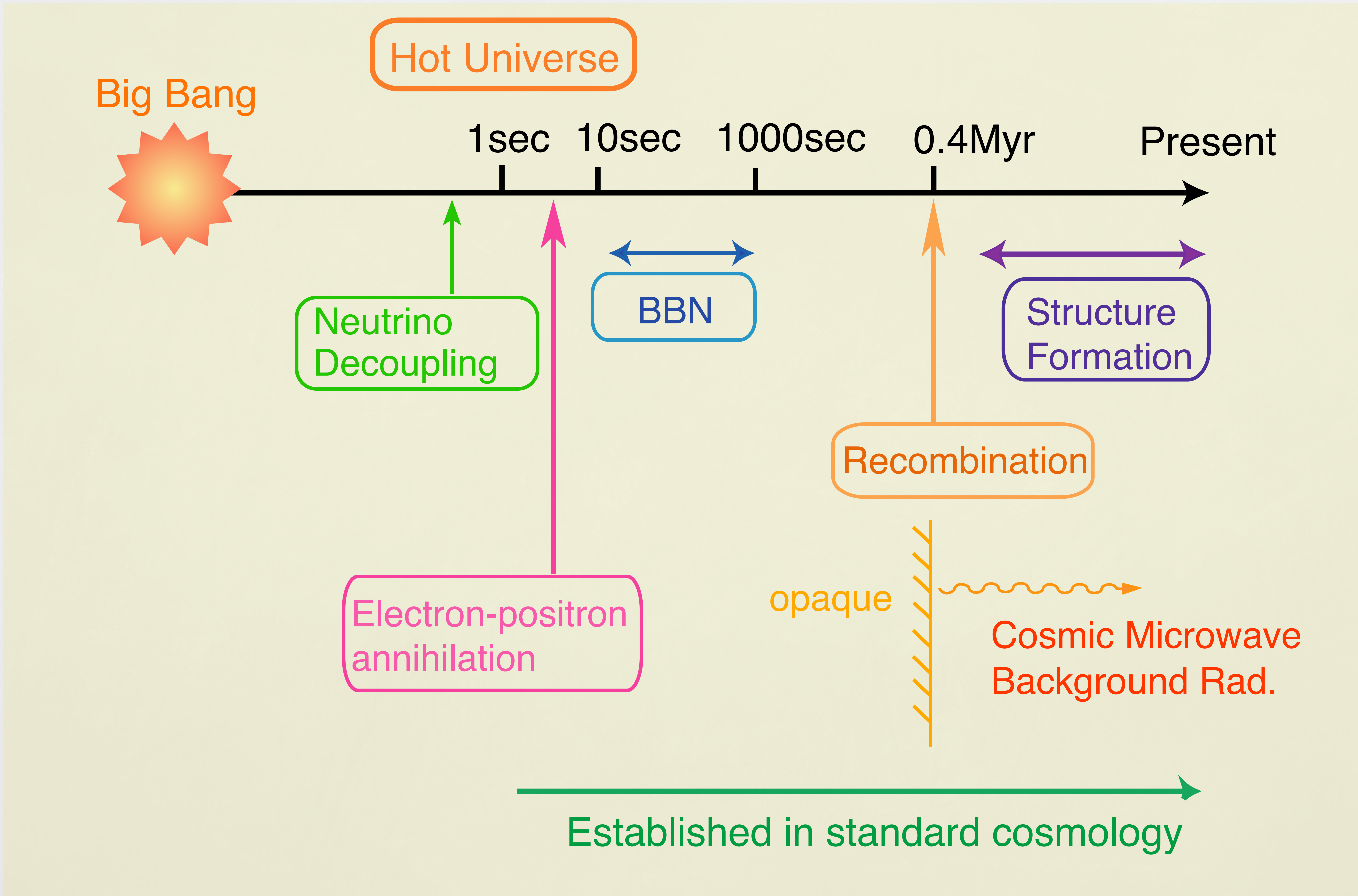
$$\Rightarrow \quad \ell_H(t) = \frac{t}{1-m} = \begin{cases} 2t & (RD) \\ 3t & (MD) \end{cases}$$

- Hubble Radius

$$\boxed{\equiv H^{-1}(t) = \frac{a}{\dot{a}}}$$

$$a \propto t^m \;\Rightarrow\; H(t)^{-1} = \frac{t}{m} = \left\{ \begin{array}{rc} 2t & (RD) \\ 3t/2 & (MD) \end{array} \right. \sim \ell_H$$

9. History of the Universe



10. Neutrino Decoupling

- T > 2 MeV
  Neutrinos are in thermal equilibrium via
  weak interaction $\nu_i + \nu_i \;\leftrightarrow\; e^+ + e^-$

  cross section: $\langle \sigma v \rangle \simeq \frac{4G_F^2}{9\pi}\langle E^2 \rangle \simeq \frac{4G_F^2}{9\pi}T^2$

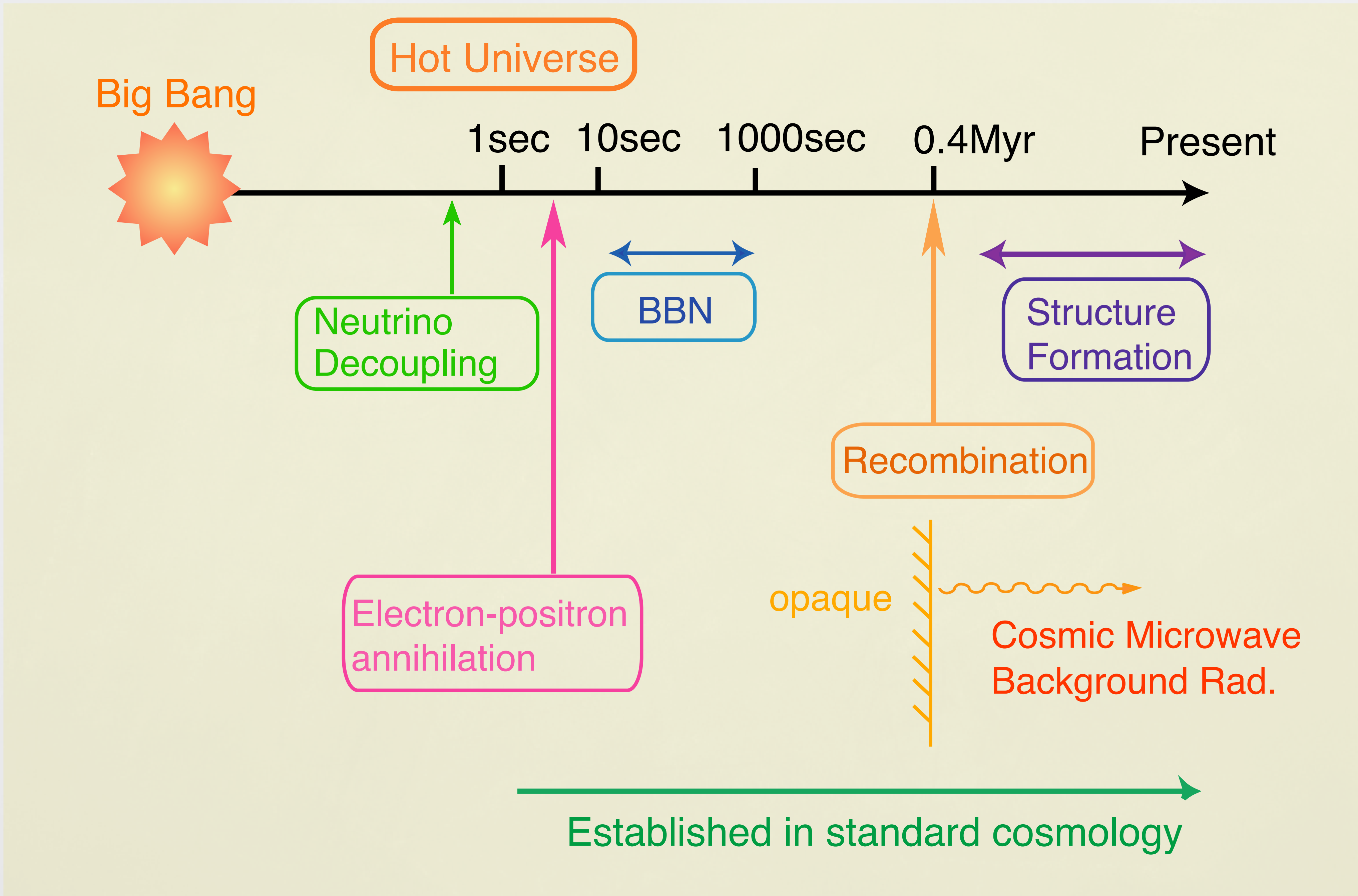
  Boltzmann eq. $\frac{dn_\nu}{dt} + 3\frac{\dot{a}}{a}n_\nu = \langle \sigma v \rangle (n_\nu^2 - n_{\nu,eq}^2)$

  $3(\dot{a}/a) \gg \langle \sigma v \rangle n_\nu \quad \Rightarrow \quad \nu \text{ decouple}$

  $3(\dot{a}/a) \simeq \langle \sigma v \rangle n_\nu \quad \Rightarrow \quad G_F^2 T^2 T^3 \simeq T^2/M_G$

  $T_d \simeq 2\ \mathrm{MeV}$

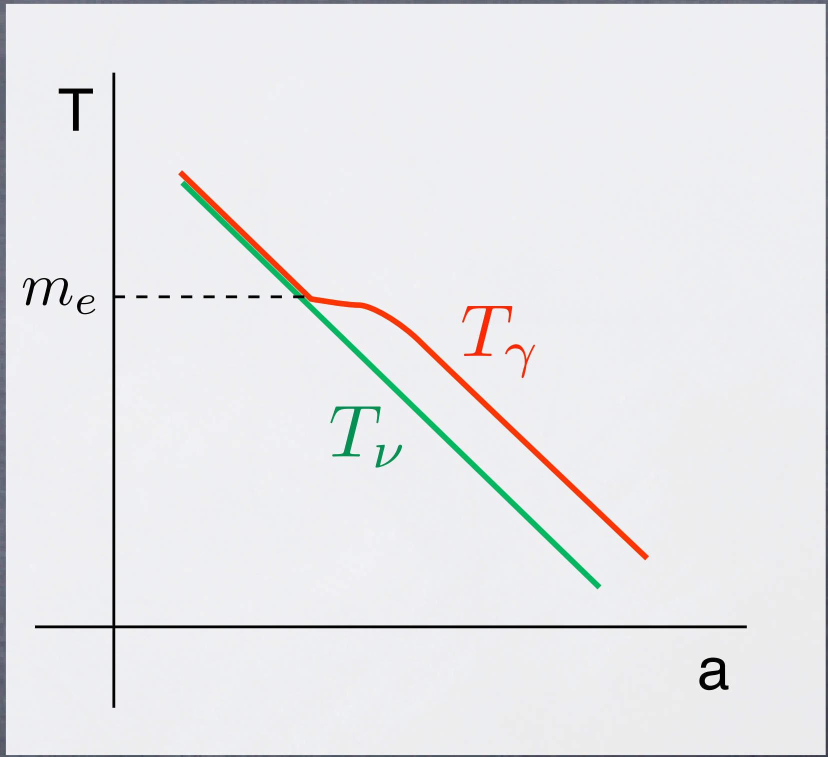
- T < 2 MeV $\boxed{e, \gamma} \not\leftrightarrow \boxed{3\nu}$


Hot Universe
Big Bang
1sec
10sec
1000sec
0.4Myr
Present
Neutrino
Decoupling
BBN
Structure
Formation
Recombination
opaque
Electron-positron
annihilation
Cosmic Microwave
Background Rad.
Established in standard cosmology

- $T_1 > m_e$ $\quad S_\gamma = a_1^3 T_1^3 \left(2 + \frac{7}{8} \times 2 \times 2\right) \frac{2\pi^2}{45}$

- $T \sim m_e$ $\quad \boxed{e^+ + e^- \rightarrow 2\gamma}$

- $T_2 < m_e$ $\quad S_\gamma = a_2^3 T_2^3 \,(2)\, \frac{2\pi^2}{45}$



Entropy conservation

$$\Rightarrow \quad T_2 = T_1 \left(\frac{a_1}{a_2}\right) \left(\frac{11}{4}\right)^{1/3}$$

On the other hand

$$T_{\nu,2} = T_{\nu,1} \left(\frac{a_1}{a_2}\right)$$

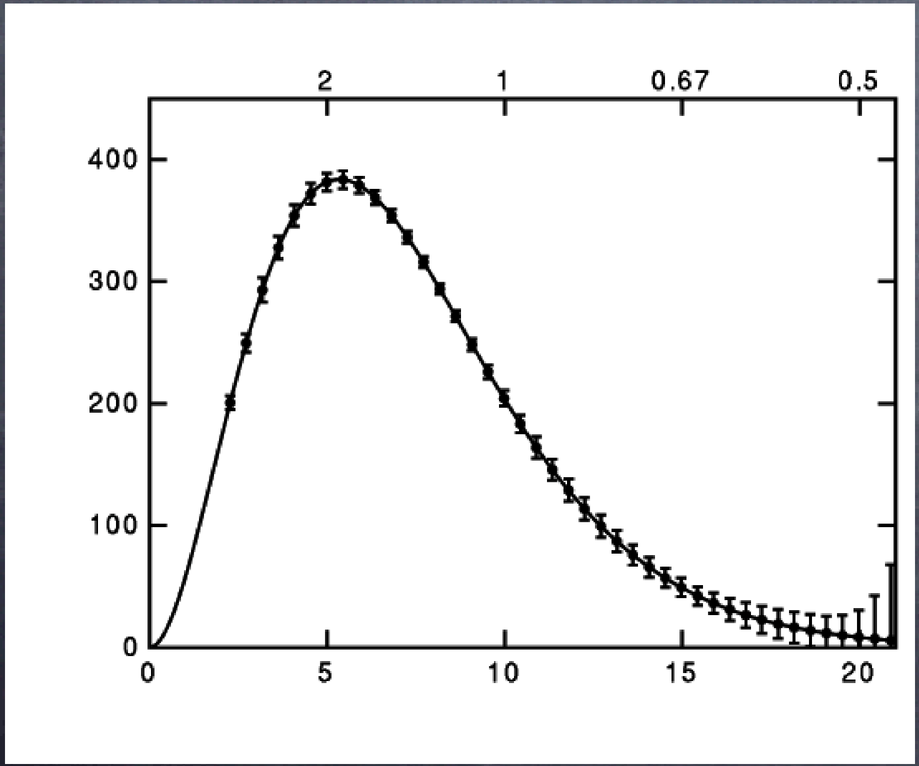
$$\Rightarrow \quad \boxed{T_\nu = T_\gamma \left(\frac{4}{11}\right)^{1/3}}$$

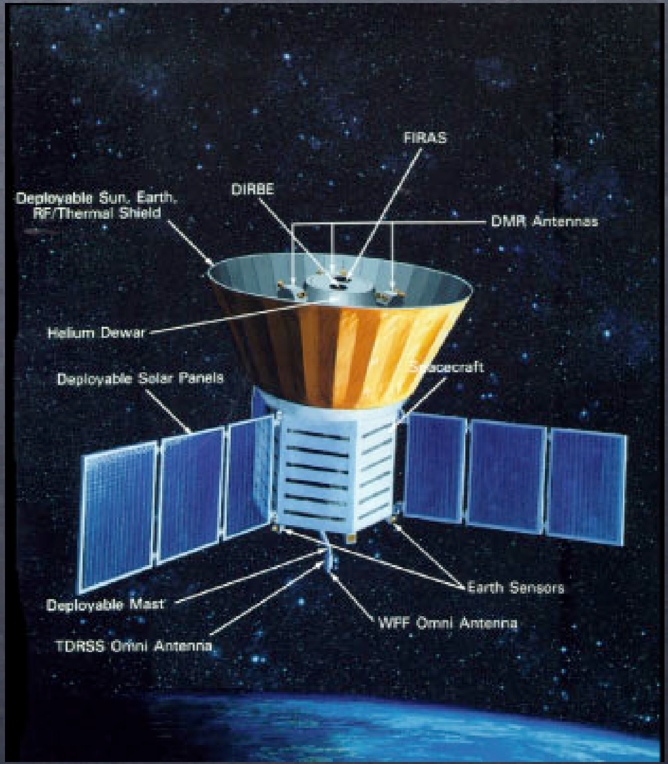
At present

$$T_{\gamma,0} = 2.726\text{K} \quad \Rightarrow \quad T_{\nu,0} = 1.95\text{K}$$

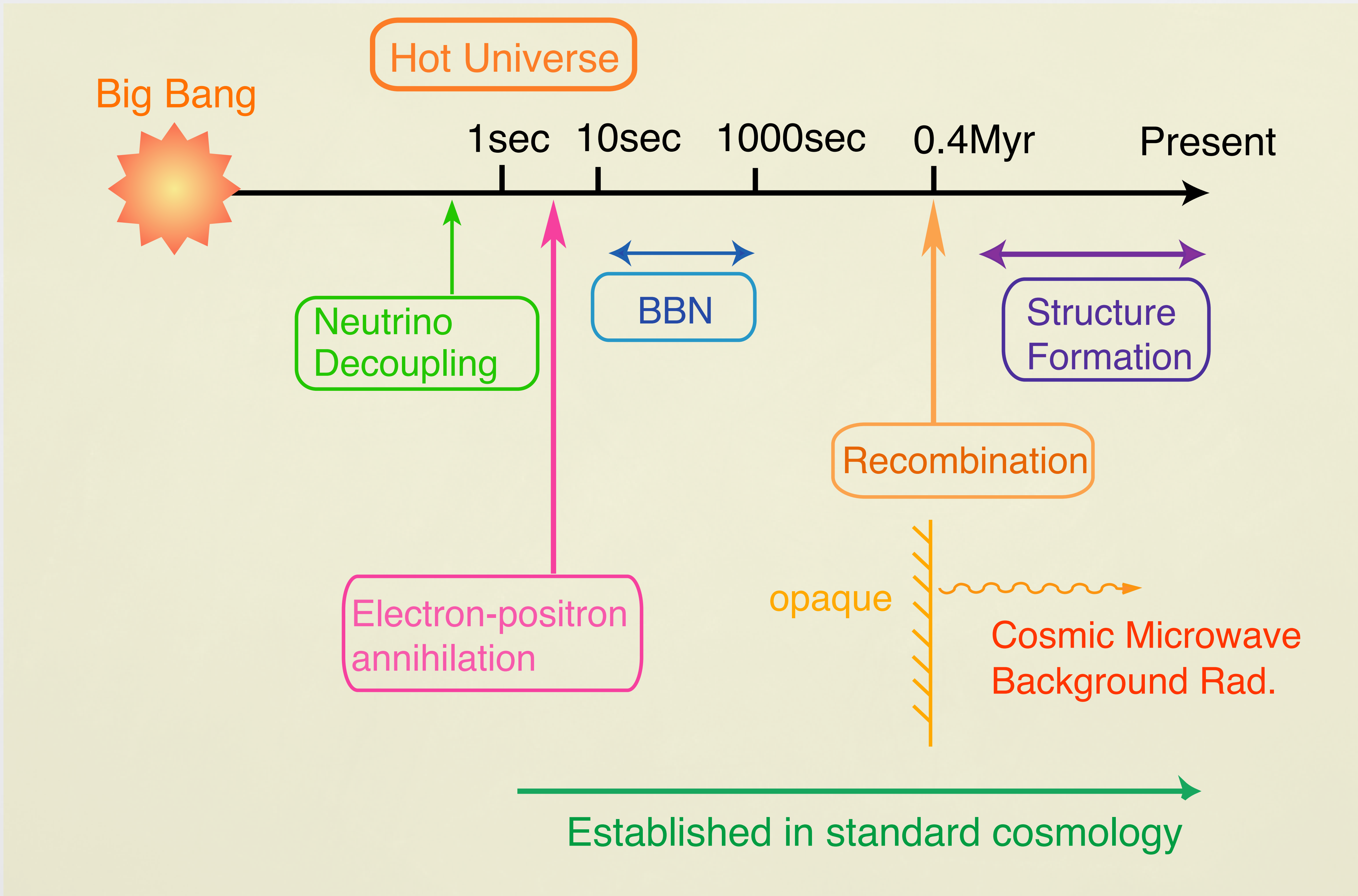
$$\frac{n_\nu}{n_\gamma} = \frac{3}{4}\left(\frac{T_\nu}{T_\gamma}\right)^3 = \frac{3}{4}\frac{4}{11} = \frac{3}{11}$$

$$n_{\gamma,0} = 415\ \text{cm}^{-3} \quad \Rightarrow \quad n_{\nu,0} = 113\ \text{cm}^{-3}$$





History of the Universe



11. Big Bang Nucleosynthesis (BBN)

In the early universe (T=1 - 0.01MeV)

$$\boxed{2p + 2n \rightarrow {}^4\mathrm{He}}$$ + small D $\;{}^3\mathrm{He}$ $\;{}^7\mathrm{Li}$

A. Initial Condition

p and n interchange via weak interaction

$$\boxed{\begin{array}{rcl} \nu_e + n & \leftrightarrow & p + e^- \\ e^+ + n & \leftrightarrow & p + \bar{\nu}_e \\ n & \leftrightarrow & p + e^- + \bar{\nu}_e \end{array}}$$

Reaction Rate $\Gamma \sim \sigma v n_e \sim G_F^2 T^2 T^3 \sim G_F^2 T^5$

$\Gamma \gg H$ ⇨ Chemical Equilibrium

$$\boxed{\mu_{\nu_e} + \mu_n = \mu_p + \mu_{e^-}}$$

$$n = \frac{g}{2\pi^2} \int_0^\infty p^2 dp \frac{1}{\exp[(E-\mu)/T] \pm 1}$$

non-relativistic

$$n = g \left(\frac{mT}{2\pi} \right)^{3/2} \exp[-(m-\mu)/T]$$

$\mu_e/T \sim 10^{-10} \ll 1$

$\mu_\nu/T \ll 1$ ⟵ assumption

$$\boxed{\mu_n = \mu_p}$$

$$\frac{n_n}{n_p} = \exp[(m_p - m_n + \mu_n - \mu_p)/T]$$
$$= \exp[-Q/T + (\mu_n - \mu_p)/T]$$

$$Q = m_n - m_p = 1.293 \text{ MeV}$$

$$\boxed{\left(\frac{n_n}{n_p}\right)_{eq} = \exp\left(-\frac{Q}{T}\right)}$$

$\Gamma \sim H \quad \Rightarrow \quad T_f$ **freeze-out temp**

$$G_F^2 T_f^5 \sim \frac{T_f^2}{\sqrt{3} M_G}\left(\frac{\pi^2}{30} g_*\right)^{1/2} \Rightarrow T_f \sim 1 \text{ MeV}$$

$$\Rightarrow \boxed{\frac{n_n}{n_p} \simeq \exp\left(-\frac{Q}{T_f}\right) \simeq \frac{1}{7}}$$

B. 0.1 MeV < T < 1 MeV

$$p + n \ \leftrightarrow \ D + \gamma \qquad Q_d = 2.22 \text{ MeV}$$

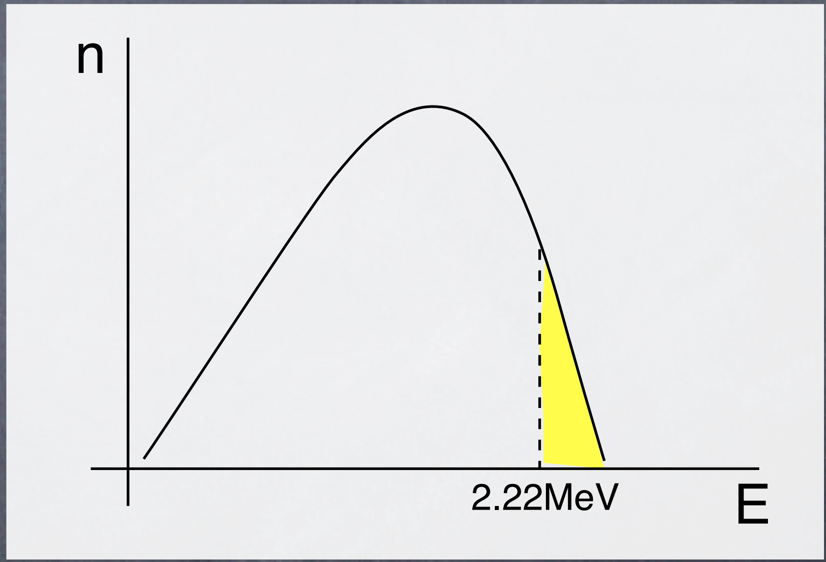
$n_\gamma \sim 10^{10} n_B \gg n_B$ **Produced D is destroyed**

$$D + \gamma \to p + n$$

$T \simeq 0.1$ MeV

$n_\gamma(E_\gamma > 2.22\text{MeV}) \searrow$

$$\Rightarrow \boxed{p + n \ \rightarrow \ D + \gamma}$$



C. T < 0.1 MeV

$$D + D \ \leftrightarrow \ {}^3\text{He} + p$$

$${}^3\text{He} + n \ \leftrightarrow \ {}^3\text{H} + p$$

$${}^3\text{H} + D \ \leftrightarrow \ {}^4\text{He} + n$$

$\Rightarrow$ ^{4}He

$+$ small amount of D, ^{3}He, ^{3}H

$({}^3\text{H} \rightarrow {}^3\text{He} + e^- + \nu_e, \quad \tau_{1/2} \sim 12\text{yr})$

C. Heavier Light Elements?

No

- No stable nuclei with A=5 or 8
- Coulomb Barrier

But tiny amount of Li7

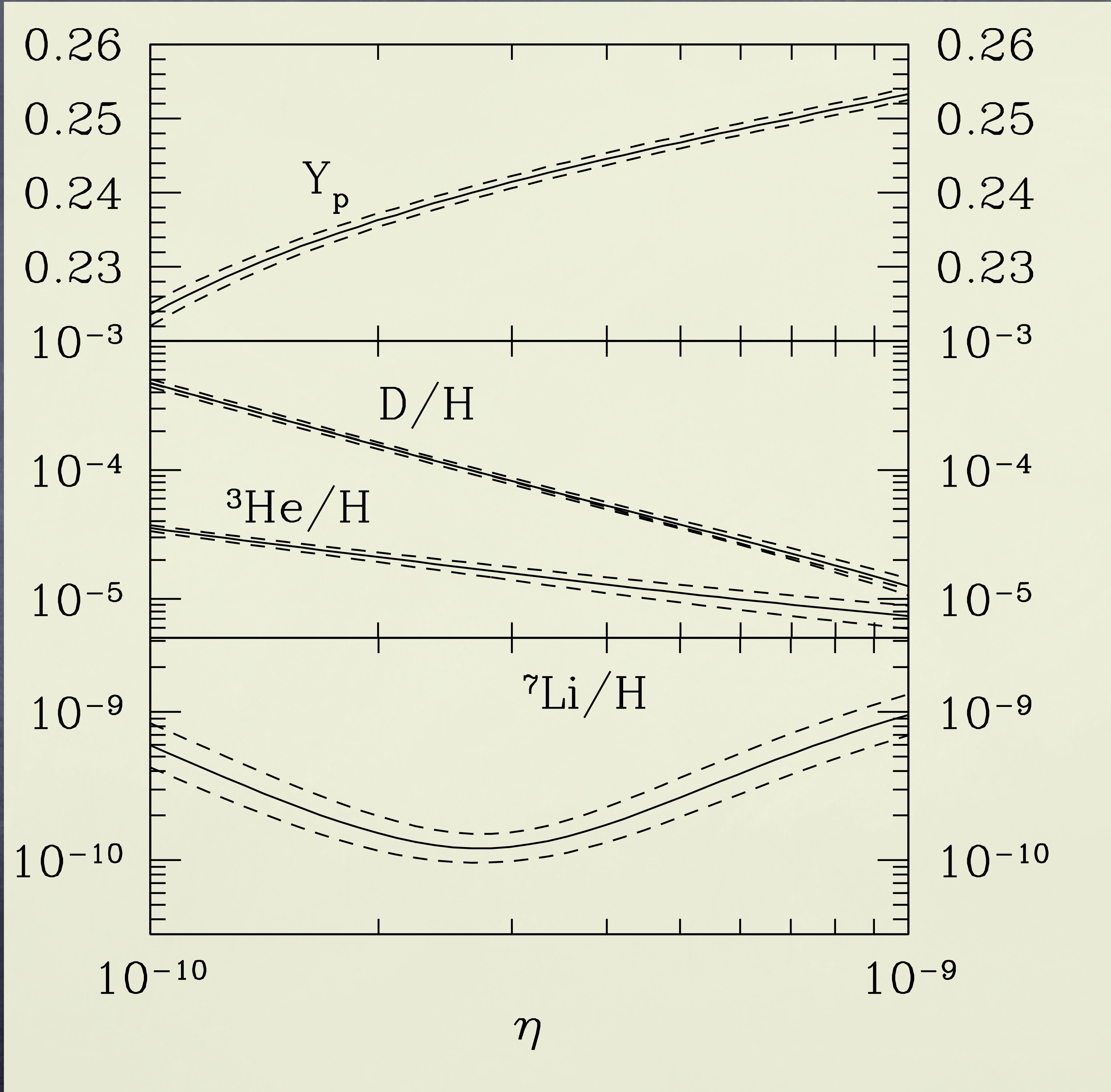
$$^4\mathrm{He} + {}^3\mathrm{H} \rightarrow {}^7\mathrm{Li} + \gamma$$

$$^4\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^7\mathrm{Be} + \gamma$$

$$\hookrightarrow {}^7\mathrm{Be} + e^- \rightarrow {}^7\mathrm{Li} + \nu_e$$

Abundances of Light Elements only depend on baryon-to-photon ratio

$$\boxed{\eta_B \equiv \frac{n_B}{n_\gamma}}$$


0.26
0.25
0.24
0.23
Y_p
10^{-3}
D/H
10^{-4}
$^3He/H$
10^{-5}
$^7Li/H$
10^{-9}
10^{-10}
10^{-10}
10^{-9}
η

Observational Abundances of Light Elements

- He4

$$Y_p = 0.238 \pm 0.002 \pm 0.005$$ Fields,Olive (1998)

$$Y_p = 0.242 \pm 0.002(\pm 0.005)$$ Izotov et al. (2003)

$$Y_p = 0.250 \pm 0.004$$ Fukugita,Kawasaki (2006)

- D/H

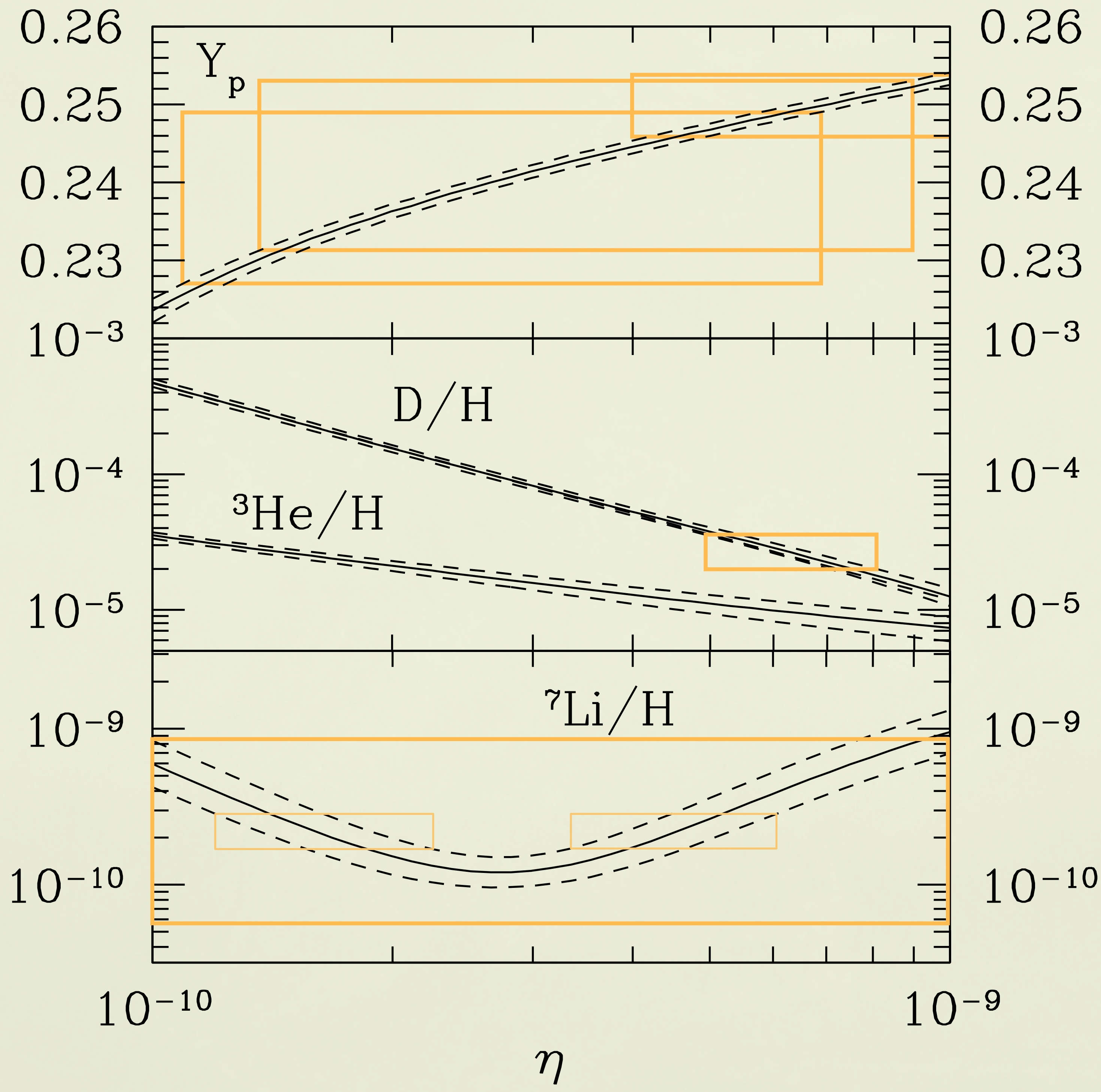
$$D/H = (2.8 \pm 0.4) \times 10^{-5}$$

Kirkman et al. (2003)

- Li7/H

$$\log_{10}(^7Li/H) = -9.66 \pm 0.056\ (\pm 0.3)$$

Bonifacio et al. (2002)



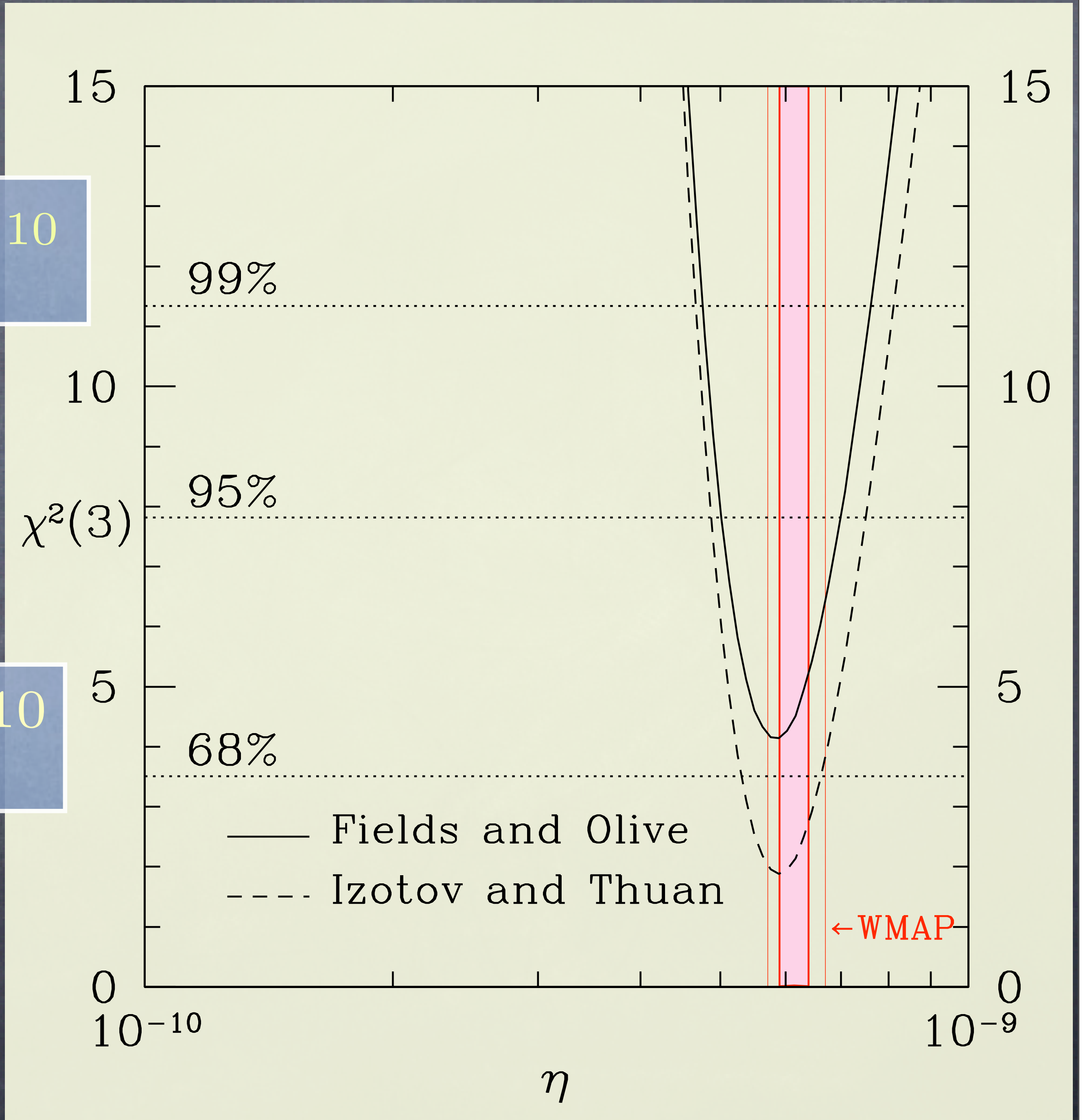
Y_p
0.26
0.25
0.24
0.23
D/H
^{3}He/H
10^{-3}
10^{-4}
10^{-5}
^{7}Li/H
10^{-9}
10^{-10}
10^{-10}
10^{-9}
η

BBN and WMAP

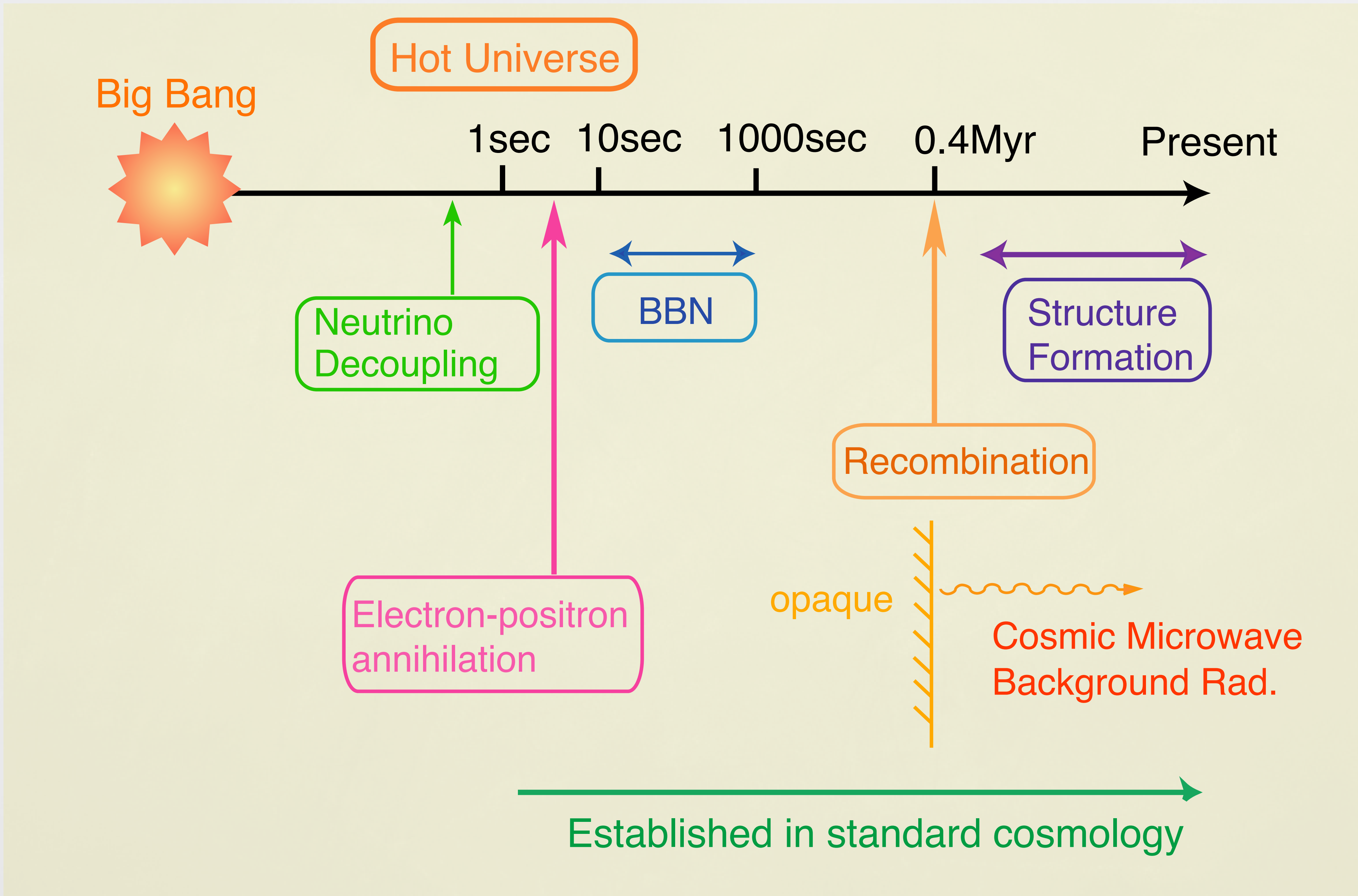
$$\eta = (5.8 - 7.5) \times 10^{-10}$$

WMAP Result

$$\eta = 6.1^{+0.3}_{-0.2} \times 10^{-10}$$



History of the Universe



12. Recombination

$$\boxed{p + e \ \rightarrow \ \mathrm{H} + \gamma}$$

at T = 4000K

ignoring He4

$$n_B = n_p + n_H \qquad n_p = n_e$$

Thermal density

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right) \quad (i = e, p, \mathrm{H})$$

Chemical equilibrium $\boxed{\mu_\mathrm{H} = \mu_e + \mu_p}$

$$\Rightarrow \quad \frac{n_\mathrm{H}}{n_p n_e} = \frac{g_\mathrm{H}}{g_p g_e} \left(\frac{m_e T}{2\pi}\right)^{-3/2} \exp\left(\frac{B}{T}\right)$$

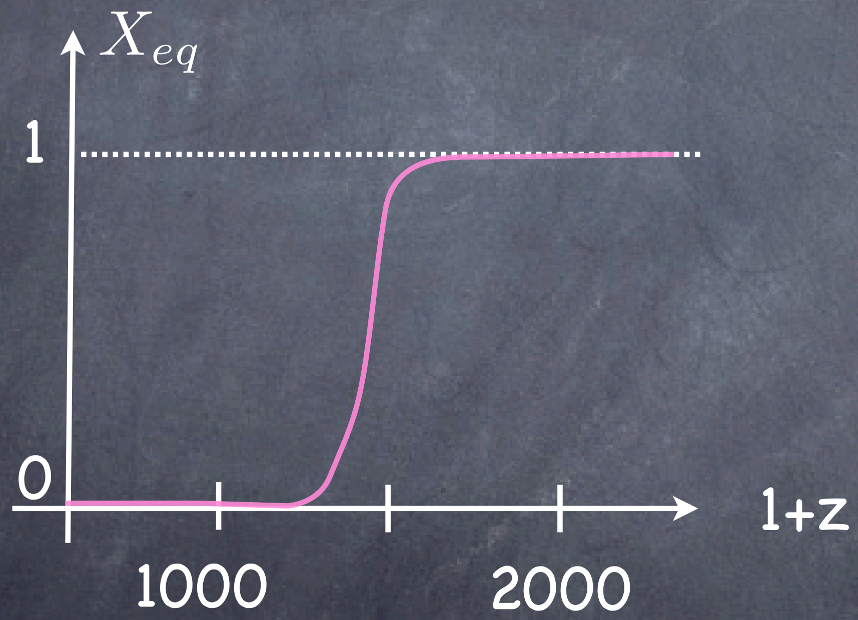
$$B = m_p + m_e - m_\mathrm{H} = 13.6 \text{ eV}$$

$$g_p = g_e = 2, \quad g_\mathrm{H} = 4$$

Define ionization fraction $X \equiv \frac{n_p}{n_B}$

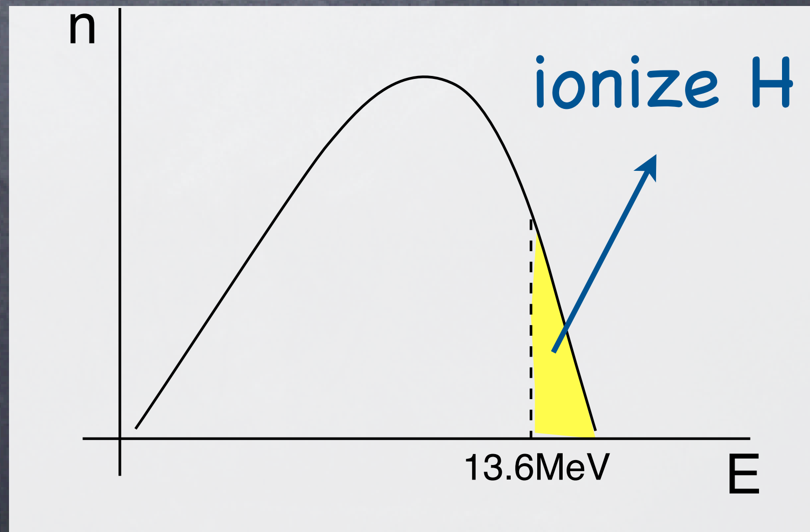
$$n_B = \eta_B n_\gamma = \eta_B \frac{2\zeta(3)}{\pi^2} T^3$$

$$\boxed{\frac{1 - X_{eq}}{X_{eq}^2} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \eta_B \left(\frac{T}{m_e}\right)^{3/2} \exp\left(\frac{B}{T}\right) \quad \text{Saha formula}}$$

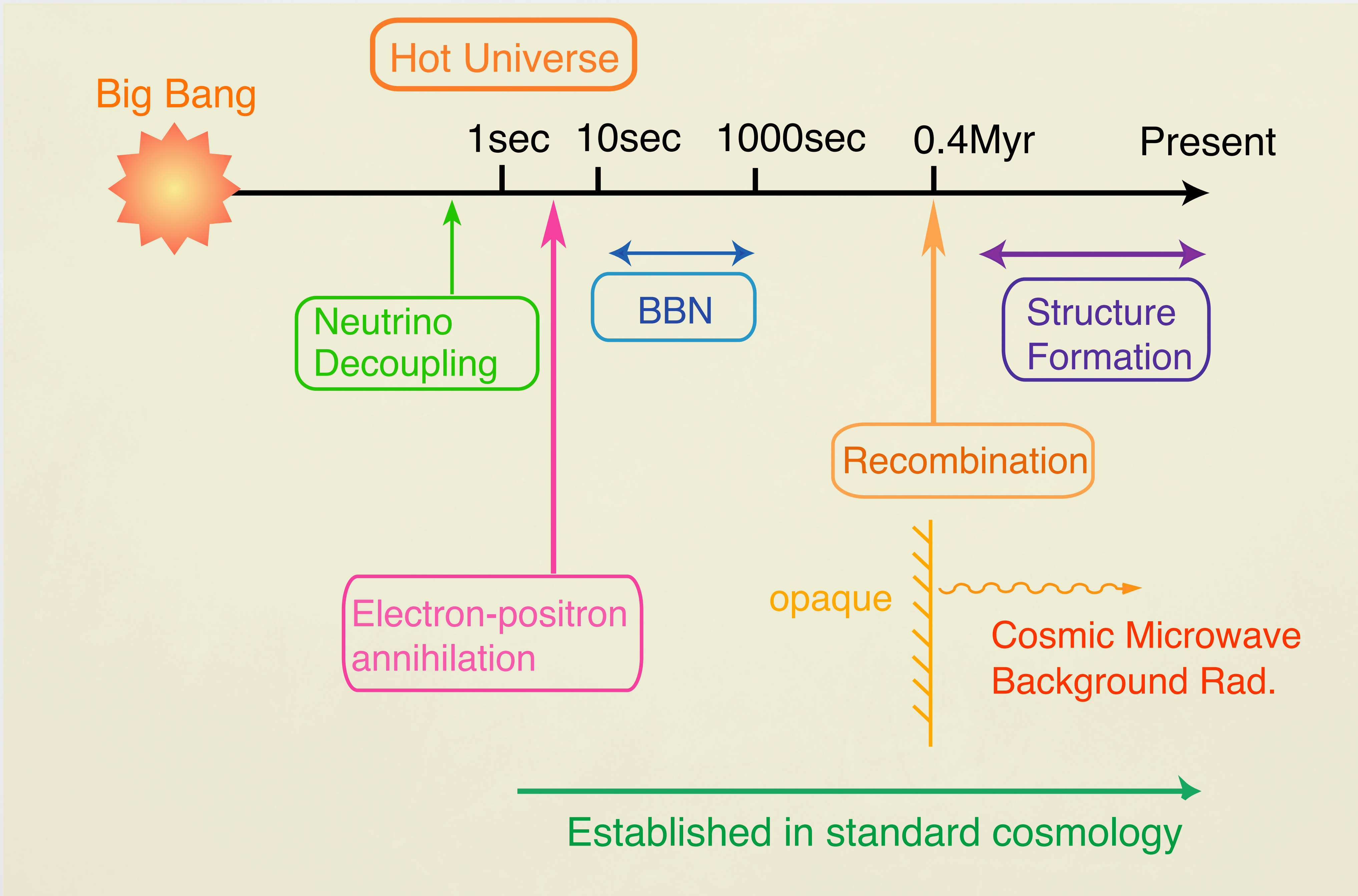


$$T_{\rm rec} \simeq 3000\ \text{K} = 0.3\ \text{eV}$$

$$T_{\rm rec} \ll B = 13.6\text{eV}$$



History of the Universe



WMAP Full Sky Map