Topological String on Calabi-Yau backgrounds

Mirror Symmetry I
AMS 2003

Symmetries of String compactifications:

Closed string comp: Example Type II / heterotic

on $M_4 \times \underbrace{CY_3}_{M}$ $\Rightarrow$ 4d $N=2$ / $N=1$

sasy in Minkowski space M_4

Motivation: 1) c_g (superstring) $= -15$ $\Rightarrow$ need

$d_{\mathbb{R}} = 6$ $(d_c = 3)$ internal manifold to cancel

Weyl anomaly

2) Start with "more unique" supergravities

in $d=11$ or $d=10$ sugra and use alg. geometry to learn about effect action in 4d

T-duality: $X: \Sigma_{g,n} \longrightarrow M$ $\quad \langle \Pi V_i \rangle = \int \mathcal{D}h \mathcal{D}X \, e^{-S} \Pi V_i$

$\downarrow$

finite dim integral $\sum_g \int_{M_{g,n}} dM_{g,n}$

$$S = \frac{1}{4\pi\alpha'} \int d\sigma^2 (g_{IJ} + b_{IJ}) \partial_\alpha X^I \partial^\alpha X^J - 2\pi \int d\sigma^2 \phi R^{(2)}$$

E.g. $M = S^1_R$ $\quad$ R = radius

Partition function $Z(R,\tau) = \langle 0 \rangle^{\text{unintegrated}}_{g=1}$

free theory with $X \sim X + 2\pi R$ boundary cond.

$$\mathrm{Tr}_{\mathcal{H}}\, q^{H_L} \bar{q}^{H_L} = \underset{\substack{\uparrow \\ \text{modulus} \\ R}}{Z(R,\tau)} = \frac{R}{l_s \sqrt{\tau_2}\, \underset{\substack{\uparrow \\ \text{oscillators}}}{|\eta|^2}} \sum_{\tilde{m},\tilde{n} \in \mathbb{Z}} \underset{\substack{\uparrow \\ \text{WS instantons}}}{e^{-\frac{\pi R}{\tau_2 l_s} |\tilde{m} - \tilde{n}\tau|^2}}$$

$R \in \mathbb{R}_+$

τ complex structure of T^2

- $\Gamma_0 = PSL(2,\mathbb{Z})$ WS repar. inv

$T: \tau \to \tau+1$ $\quad \tau_2 = \mathrm{Im}(\tau)$ inv $\quad$ Dedekind η-funct $\quad \eta(\tau+1) = e^{\frac{2\pi i}{24}}\eta$

$\sum'$ inv

$S: \tau \to -\frac{1}{\tau}$ $\quad \mathrm{Im}(-\frac{1}{\tau}) = -\frac{\tau_2}{|\tau|^2}$ $\quad \eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau) \Rightarrow \sqrt{\tau_2}\,|\eta|^2$ inv

$\sum'$ Poisson ressummation $\Rightarrow$ invariant

Poisson ressumation

$$Z(R,\tau) = \sum'_{p_L, p_R \in \Gamma_{1,1}} \frac{q^{\frac{p_L^2}{2}}\,\bar{q}^{\frac{p_R^2}{2}}}{|\eta|^2} \qquad q = e^{2\pi i\tau} \qquad (2)$$

momenta $\quad$ winding

$$p_{L/R} = \frac{1}{\sqrt{2}}\left(n\frac{l_s}{R} \pm m\frac{R}{l_s}\right)$$

$\text{mass} = \frac{1}{2}(p_L^2 + p_R^2) + \text{osc}$ $\quad \text{spin} = \frac{1}{2}(p_L^2 - p_R^2) = n\cdot m + \text{osc}$

R-duality: $\quad \Theta(1,1,\mathbb{Z})$ $\quad$ E.g. $m \leftrightarrow n$ $\quad R \to \frac{l_s^2}{R}$ $\quad \begin{matrix} p_L \to p_L \\ p_R \to -p_R \end{matrix}$

$Z(R) = Z(\frac{l_s^2}{R})$ $\quad \Rightarrow \{R > l_s\} =$

Chiral $U(1)$ gauge symmetry $\quad J^3 = i\partial X$ $\quad (1,0)$ - current $\quad \Theta(1,1,\mathbb{Z}) \backslash \Theta(1,1)/\Theta(1)\Theta(1)$

$R = l_s$ $\quad \Delta = \frac{1}{4}(n+m) \Rightarrow m = n = \pm 1$ $\quad$ two new $(1,0)$

(3) $\quad V_{mn} = :\exp(ipX + i\bar{p}\bar{X}):$ $\quad$ current $\quad J^{\pm}(z) = \frac{1}{\sqrt{2}} :e^{\pm i\sqrt{2}X}:$ $\quad SU(2)$ gauge symmetry

Mirror symmetry $\quad$ R-duality on half the dim. of compl. manifold.

Example $\quad M = T2 = \mathbb{C}/\{e_1, e_2\}$

$e_2 = e^{i\theta}R_2 \qquad e_1 = R_1 \qquad e_i \cdot e_j = g_{ij}$

$A = R_1 R_2 \sin\theta$ $\qquad e_j\, e^{*i} = \delta_j^i$

$$p_{L/R} = \frac{1}{\sqrt{2}}\left[n_i + m^j (b \pm g)_{ji}\right] e^{*i} \qquad (p_L, p_R) \in \Gamma_{2,2}$$

$$b = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

Complex structure parameter $\tau = \frac{e_2}{e_1} \in \mathbb{H} = SU(1,1)/U(1)$

Complexified volume $t = b + iA \in \mathbb{H} = SU(1,1)/U(1)$

Moduli space

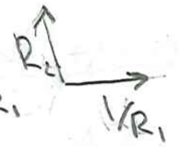
$$O(2,2,\mathbb{Z}) \backslash O(2,2,\mathbb{R}) \,/\, O(2)\times O(2) = \left(\mathbb{H}/\Gamma_0 \times \mathbb{H}/\Gamma_0 \right) / \mathbb{Z}_2^{MS} \times \mathbb{Z}_2^{WP}$$

World sheet parity $\quad m_1 \to -m_1,\ m_2 \to -m_2 \quad$ geometrically $\quad b = -b,\ p_L \leftrightarrow p_R$

Mirror symmetry $\quad n_1 \leftrightarrow m_1 \quad$ geometrically

Simplification $\quad \theta = \frac{\pi}{2} \quad R_1 \to \frac{l_s^2}{R_1} \quad b = 0$

$t = iR_1R_2 \quad \tau = i\frac{R_2}{R_1} \qquad t' = i\frac{R_2}{R_1} \quad \tau = iR_2 \cdot R_1$



Exchange of complex and compl. Kähler structure

generalize (2). It is WS rep. inv. (modular)

iff Γ_{pq} even and self dual $\Rightarrow p - q \in 8\mathbb{Z}$

Heterotic string on T4 $\quad \Gamma_{4,4+16} \quad p = 4 \quad q = 20$

$\uparrow$ $E_8 \times E_8$

Moduli space $\quad \Gamma_{pq} \otimes_{\mathbb{Z}} \mathbb{R} \cong W \oplus W^* \oplus V$

W: light like p-plane, momenta; W^*: " ", windings; V: time like plane $\cong \mathbb{R}^{q-p}$

Linear maps $\quad \phi : W \to W^* \qquad \phi : W \times W \Rightarrow \mathbb{R}$ ($g + b$)

$A : W \to V \qquad A$ Wilson lines

Space of spacelike p-planes $\quad \Pi = \{ X + \phi(x) + A(x) \mid x \in W \} \supset p_L$

$$\mathcal{M} = O(p,q,\mathbb{Z}) \backslash O(p,q) \,/\, O(p) \times O(q)$$

Aspinwall K3 review

Similar as geometric moduli space of Ricci-flat (Kähler-Einstein) metric on K3 $\quad \delta g_{i\bar{j}} \cong H_{11}(K3)$

$H_2^+ \oplus H_2^-$ $\quad 3 \quad 20$
$3 \quad 19$

Σ

$$\mathcal{M}_g = O(3,19,\mathbb{Z}) \backslash O(3,19) / O(3) \times O(19) \times \mathbb{R}_+ \qquad \dim(\mathcal{M}_g) = 58$$

Type IIA strings moduli space

+ 22 2-form b-field $\qquad \dim(\mathcal{M}_g) = 80$

$$\mathcal{M}_S = O(4,20,\mathbb{Z}) \backslash O(4,20) / O(4) \times O(20)$$

$\cup$

$\mathcal{M}_S$ exchanging $\Lambda_{Pic} \Leftrightarrow \Lambda_{trans}$
for algebraic K3

Gauge theory enhancement

IIA odd forms $C_1, C_3 \curvearrowright$ K3-2form, $dC_3^* = *_4 dC_3$
$U(1)^{24}$ $\quad 1 \quad 22 \quad 1$

het 20 right tori + $b_{\mu\nu} \curvearrowright$ T4-1form
$U(1)^{24}$ $\quad H_i \quad 4$

(0,1) chiral currents $\overset{(3)}{\curvearrowright} P_L = 0 \qquad P_R \cdot P_R = -2$

$P_R = \alpha \qquad E_{\pm\alpha_j}$

$\{\alpha_j\} \in \Gamma_{4,20} \cap \Pi^\perp \quad \alpha_j^2 = -2$

Geometrically $\alpha \rightsquigarrow$ curves C by adjunction

$C^2 = 2(g-1) \qquad C$ rational curve $\quad C^2 = -2$

$\{C_i\} \subset \Gamma_{3,19} \cap \Sigma^\perp \Rightarrow$ ADE singularity on K3

$C_i C_j = -$ Cartan-matrix of ADE

$E_{\pm\alpha_j}$ gauge bosons from D2 branes wrapping

Conjecture $\quad$ Strongly coupled IIA on K3 $\cong$ weakly coupled het String

$g_s \sim e^{\Phi} \qquad \phi_A = -\phi_{het}$

$g_A = e^{2\phi_{het}} g_{het}$

(5)

Weak – strong string duality difficult to check except for BPS saturated states.

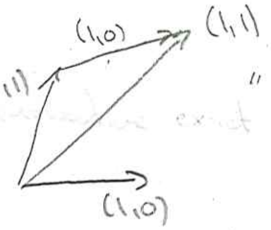
ST susy: $I = 1,2$

$$\{ Q^I_\alpha, Q^J_\beta \} = Z^{IJ} \epsilon_{\alpha\beta} \qquad Z^{IJ} = -Z^{JI}$$

$$\{ Q^I_\alpha, (Q^J_\beta)^\dagger \} = 2M \delta^I_J \delta^\beta_\alpha$$

for any particle. $M_{\vec\Gamma} \geq |Z_{\vec\Gamma}| = |\vec\Pi(t) \cdot \vec\Gamma|$ "

$\uparrow$ charge in lattice



BPS particle: $M_\Gamma = |Z_\Gamma(t)|$ "non-perturbative exact"

– charge conservation BPS particle "with primitive charge" cannot decay as long as charge lattice has full rank.

– $M_\Gamma + \Delta_{\text{non-per}} > |Z_\Gamma|$ would change deg. of freedom

Are the multiplicities of BPS particles on both sides of the String dualities the same?

Heterotic side BPS index

$$I = \mathrm{Tr}_{\mathcal{H}} (-1)^{F_L} \, q^{H_L} \bar{q}^{\bar{H}_R}$$

$\uparrow$ projects on groundstate. on the susy side, by level matching only oscillator appear on the right

η-function α_{-1} mass 1

$\alpha_{-1}^2, \alpha_{-2}$ 2

$\alpha_{-1}^3, \alpha_{-2}\alpha_{-1}, \alpha_{-3}$ 3

$$= \frac{1}{\prod_{i=1}^{\infty}(1-q^n)^{24}} = \frac{q}{\eta^{24}(q)} = \sum a_n q^n$$

$$= 1 + 24q + 324q^2 + \ldots$$

Know $C \cdot C = -\text{mass}$ $g = 2(g-1)$

BPS states must come from higher genus curves of K3.

YZ conjecture I "counts" genus g curves with g nodes. Euler # in moduli space of stable pairs. KMPS: JAMS 2010/1013

Black holes Strominger + Vafa " Morrison
Besse Einstein Manifolds "
Candelas " Trieste lectures "
Font & Theisen "Intro to String Comp."

Calabi-Yau d-fold:

M CY $\quad$ M d_c-dim Kähler manifold &

One of the followings holds

a) $c_1(TM) = 0$

b) $\exists\, g \quad R_{i\bar{j}}(g) = 0$ hard

c) $h^{d0} = 1$ & $\exists$ nowhere vanishing hom. $(d,0)$-form Ω

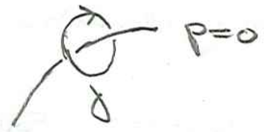
d) $\mathrm{Hol}(M) = SU(d)$

e) $\exists$ covariant const spinors $\eta\bar{\eta}$ lengthy $= \frac{1}{4}$ Sa...

establish

c) $P = 0$ Hypersurface in $\mathbb{P}^4[\omega]$ $\quad$ $z_i \to \lambda^{\omega_i} z_i \quad i = 0 \ldots 4$ defines equivalence classes $[z]$

$$P(\lambda z) = \lambda^d z$$



$$\Omega = \frac{1}{2\pi i} \int_\gamma \frac{\mu}{P}$$

$$\mu = \sum_{i=0}^{4} (-1)^i \omega_i z^i \, dz^0 \wedge \ldots \wedge d\hat{z}^i \wedge \ldots \wedge dz^4$$

well defined on $[z]$ if $\quad d = \sum_{i=0}^{4} \omega_i$

$$\Leftrightarrow \quad c(TM) = \frac{c(T_{\mathbb{P}^4})}{c(N)} = \frac{\prod (1 + \omega_i H)}{1 + dH} = 1 + O(H^2)$$

to get a form on M replace one coord z_i by P use $\int_\gamma \frac{dP}{P} = 2\pi i$ $\quad$ e.g. in V_0 patch $z_0 \neq 0$

$$\Omega = \frac{\omega_0 z^0 \, dz_i \wedge dz_j \wedge dz_k}{\Delta_0^{ijk}}$$

$$\Delta_0^{ijk} = \left| \frac{\partial(z^i, z^j, z^k, P)}{\partial(z_1 z_2 z_3 z_4)} \right|$$

$h^{00}=1$; $h^{p0} = 0$ $\quad R_{i\bar{j}} = 0 = R_{i\bar{j}k\bar{l}} = 0$

$\Delta\omega = 0 \Leftrightarrow \nabla^\nu \nabla_\nu \omega_{i_1 \dots i_p} = 0$. If M compact

$\uparrow$

Fale formula
Besse $\quad \Rightarrow \nabla_j \omega_{i_1 \dots i_p} = 0 \Rightarrow \omega$ covariant constant

on a singlet under $Hol(M)$

$\Delta \omega_{\mu_1 \dots \mu_p} = -\nabla^\mu \nabla_\nu \omega_{\mu_1 \dots \mu_p}$ if $Hol(M) = SU(d) \Rightarrow$ either $p=0$ or $p=d$

$- p R_{\nu[\mu_1} \omega^\nu{}_{\mu_2 \dots \mu_p]}$

$-\frac{1}{2} p(p-1) R_{\nu\varsigma[\mu_1 \mu_2} \omega^{\nu\varsigma}{}_{\mu_3 \dots \mu_p]}$

$\Rightarrow$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h_{22} & & 0 & \\ 1 & & h_{21} & & h_{12} & & 1 \\ & 0 & & h_{11} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$\updownarrow$ Poincaré duality $\quad \chi = 2(h_{11} - h_{21})$

$\swarrow\!\!\nearrow$ Mirror symmetry

$\longleftrightarrow$ complex con

Moduli space of CY-metrics

$R_{\mu\nu}(g) = 0$ For which nontrivial δg

$$R_{\mu\nu}(g + \delta g) = 0 \qquad (3)$$

non-trivial i.e. not the effect of reparametrisation $\quad \nabla^\mu \delta g_{\mu\nu} = 0 \quad$ gauge condition

Expand (3) to linear order

$$\nabla^\varsigma \nabla_\varsigma \delta g_{\mu\nu} - 2 R_\mu{}^\varsigma{}_\nu{}^\sigma \delta g_{\varsigma\sigma} = 0$$

(i) $\delta g_{i\bar{j}} \quad R_i{}^k{}_{\bar{j}}{}^{\bar{l}} \delta g_{k\bar{l}} = 0$

$\Rightarrow \delta g_{i\bar{j}}$ harmonic $\quad i\delta g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} = \sum_{\alpha=1}^{h_{11}} t_\alpha^{\mathbb{R}} \omega^\alpha$ $\quad \in \mathbb{R}^{h_{11}}$ Kähler cone

(ii) $\nabla^\mu \nabla_\mu \delta g_{ij} - R_i{}^k{}_j{}^l \delta g_{kl} = 0 \quad \tilde{g}_{i\bar{j}} = (g + \delta g)_{i\bar{j}} \quad \int_C \tilde{\omega} > 0$

$\omega = i\tilde{g}_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$

$\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^* \quad \Leftrightarrow \Delta_{\bar{\partial}} \delta g^i = 0 \quad \int_S \tilde{\omega}^2 > 0$

$\delta g^i = \delta g^i{}_{\bar{j}} d\bar{z}^{\bar{j}} = g^{i\bar{k}} \delta g_{\bar{k}\bar{j}} d\bar{z}^{\bar{j}} \in H_{\bar{\partial}}^{(0,1)}(M, T^{10}) \quad \int_M \tilde{\omega}^3 > 0$

$$V = v^i{}_{\bar{j}}\, d\bar{z}^{\bar{j}} \frac{\partial}{\partial x^i} \longleftrightarrow \Omega_{ijk}\, v^i{}_{\bar{j}}\, d\bar{z}^{\bar{j}}\, dz^j\, dz^k = \hat{V}$$

$$H^1(M, TM) \xrightarrow{\sim} H^{21}_{\bar{\partial}}(M)$$ iff rhs harmonic then lhs harmonic

First order linear approx $\Rightarrow$ local deformations exist. Does this extends to finite deformation space?

$$\bar{\partial} \to \bar{\partial}_z = (\bar{\partial} + V(z)) \qquad \partial_z^2 = 0$$

$$\Rightarrow \bar{\partial} V(z) + \frac{1}{2}[V(z), V(z)] \Leftrightarrow \bar{\partial}\hat{V} = -\frac{1}{2}[\hat{V}, \hat{V}] \quad (4)$$

Tian & Todorov showed that (4) can be solved in $H^{21}_{\bar{\partial}}(M)$ using $\partial\bar{\partial}$-Lemma $\Rightarrow$ Complex moduli space is unobstructed and parametrized by z.

$$t_i^{\mathbb{R}} \in \mathbb{R} \qquad (i\delta g_{i\bar{j}} + b_{i\bar{j}})\, dz^i \wedge dz^{\bar{j}} = \sum_{\alpha=1}^{h^{11}} t_\alpha \omega^\alpha$$

complexify $t_i^{\mathbb{R}} \to t_i$

Mirror symmetry conjecture:

To every CY d-fold M $\exists$ a mirror manifold W such that the complex deformation space and the Complexified Kähler deformation space is exchanged $\langle \prod V_i \rangle (M, t, z) = \langle \prod V_i \rangle (W, z, t)$

Lemma: $H^{pq}(M) = H^{d-p, q}(W)$

d odd $\quad \chi(M) = -\chi(M)$

SYZ conjecture: MS is T-duality on 3 direction in CY-3 fold

Type IIA $\Leftrightarrow$ Type IIB

even Brane — odd Branes

Mathematical statement — Derived Category of coherent sheafs — Fukaya Category of special Lagrangian 3-cycles

Geometrical Corollary:

Type IIA — Type IIB

D0-brane $\longleftrightarrow$ SLAG T^3 why T^3

$\mathcal{M}_{D0} = M$ $\quad\sim\quad$ $\dim_{\mathbb{C}} \mathcal{M}_L = b_1(L)$

$\mathcal{M}_{T^3}$

$M = SLT^3$ fibration over B_3 $\Leftarrow$ $U(1)$ connection over $T^3 \cong \hat{T}^3$, deformation space B_3

$\propto$ MS is T-duality on the fibre T3

Applications of Mirror symmetry:

Moduli spaces $\Leftrightarrow$ Vev's of scalar fields

Moduli space $\Leftrightarrow$ Vev's of scalar fields

$\mathcal{N}=2$ 4d $\quad \mathcal{M}_{\mathcal{N}=2} = \mathcal{M}_{VM} \times Q_{HM}$

Vector multiplets: Special Kähler manifold (A^μ, λ, Ψ, ϕ)

Hypermultiplets: Quaternionic manifold (Ψ_Q, Φ_Q, Ψ_Q, Φ_Q)

In the absence of charged fields (perturbative Type II) $U(1)^N$ no couplings between these (except at singularities like ADE on K3)

Mirror symmetry

Type IIA M $\qquad$ Type IIB W

$\mathcal{M}^A = \mathcal{M}^A_{h_{11}} \times Q_{h_{21}+1} \longleftrightarrow \mathcal{M}^B = \mathcal{M}^B_{h_{21}} \times Q_{h_{21}+1}$

instanton correction $\quad \phi_{IIA}$ $\qquad$ no instanton correction $\quad \phi_{IIB}$

St-St $\quad M = \begin{matrix} K3 \\ \downarrow \\ \mathbb{P}^1 \end{matrix}$

het on $K3 \times T2$ $\quad \mathcal{M}^{het} = \mathcal{M}_{geom+1} \times Q_{geom}$

ϕ_{het} $\quad$ metric moduli + bundle moduli

Closed string MS calculates

exact VM dependence of

- kinetic terms of VM $\Leftrightarrow$ exact gauge coupling
- exact masses of BPS states
- certain gravitational couplings $R_+^2 F_+^{2g-2}$ in the $\mathcal{N}=2$ effective action

Open string MS calculates exact

- superpotential
- gauge kinetic terms

$$\sum x_i^5 - z^{-1/5} \prod x_i = 0 \quad \text{in } \mathbb{P}^4$$

complex moduli

Special Kähler geometry:

$$\Omega(z) \overset{\text{def}}{=} \int_\gamma \frac{\mu}{P(z)}$$

$\mathcal{M}^B_{h_{21}}$ Kähler manifold

$$e^{-K} = i \int_W \Omega \wedge \bar{\Omega}$$

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$$

triple coupling $\quad C_{ijk} = \int_W \Omega\, \partial_i \partial_j \partial_k \Omega$

Integrability cond $\quad R^l_{i\bar{k}j} = -\bar{\partial}_{\bar{k}} \Gamma^l_{ij} = G_{i\bar{k}} \delta^l_j + G_{j\bar{k}} \delta^l_i - C_{ijm} \bar{C}^{ml}_{\bar{k}}$

$$\bar{C}^{ml}_{\bar{k}} = e^{2K} G^{m\bar{m}} G^{l\bar{l}} \bar{C}_{\bar{m}\bar{l}\bar{k}} \Rightarrow \exists \text{ holomorphic}$$

Prepotential $\quad F(z) = (X^0)^2\, \mathcal{F}(t) \qquad t^i = \frac{X^i}{X_0} \qquad i = 1 \ldots h_{21},\ I = 0 \ldots h_{21}$

which determines $\quad C_{ijk} = \partial_{t_i} \partial_{t_j} \partial_{t_k} \mathcal{F}(t)$ and $K(t, \bar{t})$

Prepotential from the periods:

Introduce sympl. basis for $H_3(W, \mathbb{Z})$ $\qquad A^I \cap B_J = \delta^I_J$, rest zero

Periods

$$\Pi = \begin{pmatrix} \int_{A^I} \Omega \\ \int_{B_I} \Omega \end{pmatrix} = \begin{pmatrix} X^I \\ F_I \end{pmatrix} = X_0 \begin{pmatrix} 1 \\ t^i \\ 2\mathcal{F} - t^i \partial_{t_i} \mathcal{F} \\ \partial_{t_i} \mathcal{F} \end{pmatrix} \begin{array}{l} D0 \\ D2 \\ D6 \\ D4 \end{array} \qquad (*)$$

If α_I, β^I dual sympl. basis of $H^3(M, \mathbb{Z})$ $\qquad \int_{A^I} \alpha_J = -\int_{B_J} \beta^I = \delta_J{}^I$

$\int \alpha_J \wedge \beta^I = \delta_J^I$ $\quad \Omega = X^I \alpha_I - F_I \beta^I \Rightarrow e^{-K} = i\left(X^I \bar{F}_I + \bar{X}^I F_I\right)$

$$\int \Omega \partial_i \Omega = \int \Omega \partial_i \partial_j \Omega = 0$$

$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ $\quad H^{21}(M)$

kinetic term & Gauge Coupl

Calculate the periods:

$$\frac{\partial \Omega}{\partial z^k} = C_k(z)\Omega + \Gamma^{(k)} \subset H^{30} \oplus H^{21}$$

Basic idea

$$\partial_{z_1}^{n_1} \cdots \partial_{z_r}^{n_r} \Omega \in \bigoplus_{i=0}^{\max(n,d)} H^{(d-i,i)}(W) \leftarrow \text{finite space}$$

$n = \sum n_i$

$\Rightarrow$ relation between the derivatives $\sim$ differential equation

Dwork - Griffith reduction method. Partial integration

$$d\phi = \ldots \frac{f\, \partial_k P}{P^{r+1}}\mu - \frac{\partial_k f}{P^r}\mu \quad \text{iff} \quad \frac{\partial}{\partial x_k}\left(\frac{f(x_k)}{P^r}\mu\right) \quad \text{hom of degree zero}$$

Example Quintic: $\quad h_3(W) = 4$

$$\mathcal{L}_0 \Pi(z) = \left[\Theta^4 - 5z \prod_{m=1}^{4}(5\Theta + m)\right]\Pi(z) = 0 \qquad \Theta = z\frac{d}{dz} \qquad (5)$$

Which solution correspond to symplectic basis of cycles? Mirror symmetry predicts at some point $(z=0)$ and for some choice of X^0

$$C^A_{abc} = \int \omega_a \wedge \omega_b \wedge \omega_c + \mathcal{O}(e^{2\pi i t}) \qquad \mathcal{F} = -\underbrace{\mathcal{K}_{abc}}_{\int \omega_a \wedge \omega_b \wedge \omega_c} \frac{t^a t^b t^c}{3!} + \underbrace{A_{ab}}_{\frac{1}{2}\int \omega_a \wedge \omega_b} \frac{t^a t^b}{2} + \underbrace{c_a}_{\frac{1}{24}\int_W \omega_a \wedge c_2} t^a - i\underbrace{\chi}_{\int_W c_3(W)} \frac{\zeta(3)}{2(2\pi i)^3} + f(e^{2\pi i t}) \qquad (6)$$

corrections

instantons $\operatorname{Im} t \to \infty$ vanish

& $\exists$ x shift symmetry $t^i \to t^i + 1$ integer shift of B-field. This characterizes this point as a point of maximal unipotent monodromy. $\Pi \Rightarrow M\Pi$ $\quad (M - 1)^{\dim_{\mathbb{C}}(W)+1} = 0$

In the T2 and K3 case we had complete description of moduli space. What is the analog?

1) Local torelli theorem holds: Locally periods are good coord on the moduli space

2) Discrete group replaced by the mondromy groups $\mathbb{Z}_2^{MS} \times \left(\Gamma_{CM} \in Sp(h_3(W), \mathbb{Z})\right) \times \left(\Gamma_{K3} \subset Sp(h_3(M), \mathbb{Z})\right)$

Solutions to (5)

4 degenerate roots

$$\varpi = \sum_{n=0}^{\infty} c_n z^{n+\alpha} \qquad \text{apply } \mathcal{L}_5 \Rightarrow \alpha^4 c_0 = 0$$

$$c_n = c_{n-1} \frac{\prod_{k=0}^{4} (5n-k)}{n^5} \overset{c_0=1}{\Rightarrow} c_n = \frac{(5n)!}{(n!)^5}$$

little problem no other power serie solutions

Frobenius method

$$\varpi(z,\rho) = \sum_{n=0}^{\infty} \frac{\Gamma(5n+1+\rho)}{\Gamma(n+1+\rho)^5} z^{n+\rho} \qquad \varpi_0(z,0) \text{ Solution}$$

$[\partial_\rho, \mathcal{L}_5] = 0$ on solution space

$$X_0 = \varpi_0(z,0) = 1 + 120 z + \ldots$$

$$X_1 = \frac{1}{2\pi i} \partial_\rho \varpi(z,\rho)\big|_{\rho=0} = \frac{1}{2\pi i} X_0 \cdot \log(z) + S_1(q)$$

$$\varpi_2 = \left(\frac{1}{2\pi i}\right)^2 \partial_\rho^2 \;" \;= \left(\frac{1}{2\pi i}\right)^2 \log(z)^2 X_0 + 2 S_1(q) \log(z) + \ldots$$

$$\varpi_3 = \left(\frac{1}{2\pi i}\right)^3 \partial_\rho^3 \;" \;= \left(\frac{1}{2\pi i}\right)^3 \log(z)^2 + \ldots$$

Complexif. Area $\quad t = \frac{X^1}{X_0}(z) \quad z = 0 \quad t \to t+1 \quad$ mirror map

organize solution so that (*) holds with (5)

$\Rightarrow$ integral basis $\quad \int c_2 \wedge \omega = 50 \quad A_{11} = \frac{11}{2}$

Δ Discriminant

$$M_{z=0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & -3 & 1 & -1 \\ -8 & -5 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}_5 = z^4 (1 - 5^5 z) \partial^{(4)} + \ldots$$

at $z = \frac{1}{5^5} \quad X_i = 1 \quad P = 0 \quad \frac{\partial P}{\partial X_i} = 0$

first order $\quad z = \frac{1}{5^5} + \mu \quad X_i = 1 + \varepsilon_i \quad \delta_{l_i}$ liner in ε_i

$u_1 u_2 - u_2 u_3 = \mu \quad$ conifold : one $S_3 \searrow pt$

Lefshet monodromy in odd dim

$$C_i \to C_i - (\nu \cdot C_i) \nu$$

$$M_{z=\frac{1}{5^5}} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad M_\infty = M_0^{-1} M_{5^5}^{-1} \qquad M^B_{h_{21}(M)} = \mathbb{P}^1 \setminus \{0, 5^5, \infty\}$$

$m_\Gamma = |Z_\Gamma| = |\Gamma \cdot \Pi(z)| = -\int_M e^{-\omega}\, ch(A) \sqrt{Td(M)} + inst$

A- A_Γ D-brane in derived category of coh sheaves with K-theory charge Γ.

Instanton expansion:

$$F(z(t)) = \text{classical terms} + \sum_{\beta \in H_2(M,\mathbb{Z})}' Li_3(q^\beta)\, n_0^\beta$$

Quintic	$g=0$	$g=1$
$d=1$	2875	0
2	609 250	0
3	317 206 375	609 250

Kontsevich Givental 96 Zinger 08

<u>Construction of mirror manifolds:</u>

Quintic $P(X) = 0$ 126 monome $H^1(M, T_M) \cong H^{21}(M)$

x_i^5 (5), $x_i^4 x_j$ (20), $x_i^2 x_j^2 x_k$ (30), $x_i x_j x_k^3$ (30), $x_i x_j x_k x_l^2$ (20) $\prod_{i=0}^{4} x_i$ (1)

$GL(4)$ 25 reparametrization expect $101 = h^{21}$

$X_s = \text{Quintic}/\mathbb{Z}_5^3$ $x_i \to x^i e^{\frac{2\pi i}{5} a_i}$ $a_i \in \mathbb{Z}$ $\sum' a_i = 0 \mod 5$

$X_s\ P_s = \sum' x_i^5 - z^{-1/5} \prod x_i \subset \mathbb{P}^4/\mathbb{Z}_5^3$

resolve $h_{11}(\hat{X}_s) = 101$ $h_{21}(\hat{X}_s) = 1$

Batyrev: reflexive lattice polyhedra Δ: Convex hull of lattice points containing the origin Δ bounded by Hyperplanes with distance one from the origin

Kreuzer review 2006

$l \in \Lambda^*$ $\langle x, l \rangle = -1$
prim

Pair (Δ, Δ^*) of r.l.p $\Delta \subset \Lambda_\mathbb{R}$ $\Delta^* \subset \Lambda_\mathbb{R}^{*}$

$\Delta = \{ x \in \Lambda \otimes_\mathbb{Z} \mathbb{R} \mid \langle x, y \rangle \geq -1,\ y \in \Delta^* \}$ $\Delta^{**} = \Delta$

$\Rightarrow$ ambient space $\mathbb{P}_\Delta$, $\mathbb{P}_{\Delta^*}$

claim X_Δ X_{Δ^*} Hypersurfaces defined by canonical divisors are mirror pairs

$$h_{11}(X_\Delta) = h_{21}(X_{\Delta^*}) = l(\Delta^*) - 1 - \dim\Delta - \sum'_{\operatorname{codim}(\Theta^*)=1} l^*(\Theta^*) - \sum' l^*(\Theta^*)\, l^*(\Theta)$$

l all points of poly

l^* inner points " "

reparametrisations

Mori vectors $l^{(\alpha)}$ special basis of relations between points in Δ^*, i.e. $\sum' l_i^{(\alpha)} \nu_i^* = 0$ $\quad \nu_i^* \subset \Delta^* \cap \Lambda^*$

ν_0^* inner point

$$z^{(\alpha)} = \prod_i a_i^{l_i^{(\alpha)}} \qquad \left(\prod_{l_i^{(\alpha)}>0} \partial_{a_i}^{l_i^{(\alpha)}} - \prod_{l_i^{(\alpha)}<0} \partial_{a_i}^{l_i^{(\alpha)}} \right) \frac{\Pi}{a_0} = 0$$

$z^{(\alpha)}$ point of M.M

HKTY (93 + 94)

Higher genus amplitudes, BPS-States,

$$F(\varepsilon_1, \varepsilon_2, t) = \sum_{\substack{s=0\\ g=0}}^{\infty} (\varepsilon_1+\varepsilon_2)^{2s} (\varepsilon_1 \varepsilon_2)^{g-1} F^{(s,g)}(t)$$

$\varepsilon_1 = -\varepsilon_2 = i g_s \Rightarrow$ only $F^{(0,g)}(t)$ contributes

$$F^{(0,0)}(t) = \mathcal{F}(t) \qquad F^{(0,g)} = \text{class} + \sum' r_g^\beta q^\beta$$

$$r_g^\beta = \int_{\mathcal{M}(g,\beta)} C_{vir}(g,\beta) \in \mathbb{Q}$$

calculated by localisation, Kontsevich Givental

$\mathcal{M}(g,\beta) \leftarrow$ moduli space of maps

$$\dim_{vir} \mathcal{M}(g,\beta) = \int_\beta c_1(TM) + (g-1)(3 - \dim_{\mathbb{C}}(M)) \overset{!}{=} 0$$

for CY-3 folds $\rightsquigarrow$ all r_g^β potentially nonzero

BPS D2/D0 BPS invariants

Different perspective: Consider not moduli space of maps, but of D2/D0 branes or stable pairs

Physical expectation:

- Charge $\beta \in H_2(M,\mathbb{Z}) \Rightarrow m = \int_\beta \omega$
- Spin in $SU(2)_L \otimes SU(2)_R$ 5d little group

<u>Expect BPS multiplicity</u> $N^\beta_{j_L j_R}$ counted by

index $I = \mathrm{Tr}_R (-1)^{m_L + m_R} q_L^{m_L} q_R^{m_R} e^{-\beta \cdot t}$

$q_{L/R} = e^{i\varepsilon_{L/R}}$ $\varepsilon_{L/R} = \frac{1}{2}(\varepsilon_1 \pm \varepsilon_2)$ $m_{L/R}$ is $\sigma^3_{L/R}$ eigenvalue

Calculate amplitude $R_-^2 F_-^{\varepsilon_-^2} F_+^{\varepsilon_+^2}$

$\left.\begin{array}{l}+ \text{ selfdual} \\ - \text{ antiselfdual}\end{array}\right\}$ part of curvature 2-form and graviphoton field strength

$$F(\varepsilon_\pm, m) = -\int_\varepsilon^\infty \frac{ds}{s} \frac{\mathrm{Tr}(-1)^{\sigma_L^3 + \sigma_R^3} e^{-sm^2} q_L^{s\sigma_L^3} q_R^{s\sigma_R^3}}{4\left(\sinh^2\left(\frac{s\varepsilon_-}{2}\right) - \sinh^2\left(\frac{s\varepsilon_+}{2}\right)\right)} \quad (6')$$

$$Z = e^F = \prod_\beta \prod_{j_{L/R}}^\infty \prod_{m_{L/R} = -j_{L/R}}^{j_{L/R}} \prod_{m_{1/2}=1}^\infty \left(1 - q_R^{m_R} q_L^{m_L} e^{\varepsilon_1(m_1 - \frac{1}{2})} e^{\varepsilon_2(m_2 - \frac{1}{2})} e^{-\beta t}\right)^{(-1)^{2(j_R + j_L)} N^\beta_{j_R j_L}}$$

~~I is~~ only well defined if supersymmetry gen Q_R can be twisted by an R-sym $\Leftrightarrow$ M has an isom

$\Rightarrow M$ non-compact CY-3-fold $\mathcal{O}(-K_B) \to B$ B fano

Calculation of $F^{(n,g)}$. Holomorphic anomaly equation

Huang - Kashani-Poor

$$\partial_{\bar{\imath}} F^{(n,g)} = \frac{1}{2} \bar{C}^{jk}_{\bar{\imath}} \left(D_j D_k F^{(n,g-1)} + \sum_{m,h}' D_j F^{(m,h)} D_k F^{(n-m,g-h)} \right)$$

$n+g > 1$

$$\partial_i \bar{\partial}_{\bar{\jmath}} F^{(0,1)} = \frac{1}{2} C_{ijk} \bar{C}^{jk}_{\bar{\jmath}} - \frac{\chi - 1}{24}$$

LMS : $W \to \mathcal{C}_g$ Riemann surface

$g = 1$ $\Omega \Rightarrow \lambda$ meromorfic diff

$$F^{(n,g)} = \frac{1}{\Delta^{2(g+n)-2}(z)} \sum_{k=0}^{3g+2n-3} X^k \, p_k^{(n,g)}(z)$$

polynomial in z of degree at most $(2(g+n)-2)d_\Delta$

$$X = \frac{g_3(u)}{g_2(u)} \frac{E_4(\tau)}{E_6(\tau)} \hat{E}_2(\tau)$$

Coefficients of Weierstrass form

E_{2n} Eisenstein series $\qquad \hat{E}_2 = E_2(\tau) - \frac{3\pi i}{\mathrm{Im}(\tau)}$

polynomials $p_0^{(n,g)}$ are determined by the Conifold gap condition: $S = (\varepsilon_1 + \varepsilon_2)^2$ perform (6) for a single BPS state of mass $m = t = \int_\nu \lambda$

$$F = -\left[-\frac{1}{12} + \frac{1}{24} s g_s^{-2}\right] \log(t) +$$
$$\left[-\frac{1}{240} g_s^2 + \frac{7}{1440} s - \frac{7}{5760} s^2 g_s^{-2}\right] \frac{1}{t^2} +$$

$$\Rightarrow F^{(n,g)} = \frac{N^{(n,g)}}{t^{2(g+n)-2}} + \mathcal{O}(t^0)$$

Absence of subleading terms determines $p_0^{(n,g)}$

E.g. for $\mathcal{O}(-3) \to \mathbb{P}^2$ given g

d

$$N_{\frac{1}{4}(d-1)(d-2), \frac{1}{4}d(3+d)} = 1$$

$$N^2_{j_L j_R} = 0 \quad \text{if } (2(j_L - j_R) + d) \bmod 2 = 0$$

Topological A-model
Topological Vertex

The top. Vert. is a tool to solve
open-closed
the top A-model on toric CY manifolds
using Gauge / String theory correspondence.

Chern Simons theory → Gauge
Topolog. String → String theory

String theory on CY manifolds

$$X : \Sigma_{g,h} \longrightarrow M_d$$

M_d ← Calabi-Yau manifold

M CY: M $d_{\mathbb{C}}$-dim Kähler manifold & one of the following holds

- $c_1(TM) = 0$
- $\exists\ g$ $R_{i\bar{j}}(g) = 0$
- $h^{d,0} = 1$ & $\exists$ no where vanishin $(d,0)$ form Ω
- $\mathrm{Hol}(M) = SU(d)$
- $\exists$ 2 covariant const. spinors $\eta, \bar{\eta} \Rightarrow \frac{1}{4}$ susy comp

$M_3 \Rightarrow$ Type II $\mathcal{N}=2$ in 4d

Worldsheet theory

$\mathcal{N} = (2,2)$ Super conformal WS Theory

This WS symmetry is generated by 2 copies of

	h	Q	Name
T	2	0	Energy momentum tensor
J	1	0	$U(1)$ current
$G^{\pm}$	$3/2$	± 1	Super partners of EMT

chiral half

$$T(z)\,T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{2}\partial T$$

$$T(z)\,G^{\pm}(0) \sim \frac{3}{2z^2}G^{\pm}(0) + \frac{1}{2}\partial G^{\pm}(0)$$

$$T(z)\,J(0) \sim \frac{1}{2}z^2 J(0) + \frac{1}{z}\partial J \qquad (1)$$

$$G^{+}(z)G^{-}(0) \sim \frac{2c}{3z^3} + \frac{2}{z^2}J + \frac{2}{z}T + \frac{1}{z}\partial J$$

$$J(z)\,G^{\pm}(0) \sim \pm\frac{1}{z}G^{\pm}(0)$$

$$J(z)\,J \sim \frac{c}{3z^2}$$

Other no poles

Charges $\quad A_n = \int_{C_0} \frac{dz}{2\pi i}\, z^{n+h(A)-1} A(z) \qquad (2)$

$Q_+ \sim G_0^-$ $\quad \bar{Q}_+ \sim G_0^+$ $\quad Q_- \sim \bar{G}_0^-$ $\quad \bar{Q}_- \sim \bar{G}_0^+$

$Q_\pm^2 = \bar{Q}_\pm^2 = 0 \qquad (3)$

F_V Vector $U(1)$

F_A axial $U(1)$

$\{Q_\pm, \bar{Q}_\pm\} = H \pm P$

$[M_E, Q_\mp] = \mp Q_\pm \qquad [M_E, \bar{Q}_\pm] = \mp \bar{Q}_\pm \qquad (3)$

WS Euclidean Lorentz group

$[F_V, Q_\pm] = -Q_\pm \qquad [F_V, \bar{Q}_\pm] = \bar{Q}_\pm$

$[F_A, Q_\pm] = \mp Q_\pm \qquad [F_A, \bar{Q}_\pm] = \pm \bar{Q}_\pm$

Topological Theory

$Q_\pm$ $\bar{Q}_\pm$ nil potent operators $\quad Q^2=0$

Wants topological theory with cohomological states

$$Q|\eta\rangle = 0 \qquad |\eta\rangle \sim |\eta\rangle + Q|\lambda\rangle$$

$$H_Q = \frac{Q \text{ closed}}{Q \text{ exact}} \quad \text{finite HS of physical operators}$$

Twisting

However $Q_\pm$ $\bar{Q}_\pm$ not globally defined scalars $\lightning$

Twist A-twist $\quad \mathcal{M}_{E'} = \mathcal{M}_E + F_V$

B-twist $\quad \mathcal{M}_{E'} = \mathcal{M}_E + F_A$

	$U(1)_E$	$U(1)_A$	$U(1)_V$	spin	A-twist $U(1)_{E'}$	spin	B-twist $U(1)_{E'}$	spin
Q_-	1	1	-1	$K^{1/2}$	0	$\parallel$	2	K
$\bar{Q}_+$	-1	1	1	$\bar{K}^{1/2}$	0	$\parallel$	0	$\parallel$
$\bar{Q}_-$	1	-1	1	$K^{1/2}$	2	K	0	$\parallel$
Q_+	-1	-1	-1	$\bar{K}^{1/2}$	-2	$\bar{K}$	-2	$\bar{K}$

$$\hat{T}(z) = T(z) \pm' \tfrac{1}{2}\partial J \qquad (\hat{G}^+, J(z), \hat{T}(z), G^-) \sim (J_{BRST}, j_g, T^{mrg}, b)$$

$\Rightarrow$ $\quad Q_A = Q_- + \bar{Q}_+$ ✓ good nil potent op $\quad (+,-)$

$\quad Q_B = \bar{Q}_- + \bar{Q}_+$ ✓ " " $\quad (4) \quad (+,+)$

Remarks

1) $$\int_{\Sigma} \partial_\mu j^\mu_A = 2\int c_1(X^*(T^{10}M)) \tag{5}$$

0 only if $c_1(TM)=0$

$\Rightarrow$ B-model exists only on CY manifolds

2) $$H_{Q_A} \approx H_{\text{de Rham}}(M) \tag{6}$$

$$H_{Q_B} \approx \bigoplus_{p,q=0}^{n} H^{0,p}(M, \Lambda^q TM) \tag{7}$$

Topological - A model

$$S_A = it \int_{\Sigma} d^2z \ \{Q_A, V\} + t \int_{\Sigma'} X^*(\underset{\substack{\uparrow \\ \text{Kähler form}}}{iJ} + \overset{\substack{\text{B-field} \\ \downarrow}}{B})$$

Path integral depends only on Kähler parameters of M.

$$Z = \int \mathcal{D}X \, \mathcal{D}(\text{ferm}) \, \mathcal{D}h \; e^{S(X, h, \text{ferm}, \overbrace{B, G, \varphi}^{\text{Background data}})}$$

Critical dimension $= \sum_g \lambda^g \int_{\overline{M}_g(\Sigma')} \int \mathcal{D}X \, \mathcal{D}(\text{ferm}) \; e^{S(X, B, G, \varphi)}$

$$\lambda = e^{\phi}$$

A-twist $= \sum'_{g, \beta \in H_2(M, \mathbb{Z})} \lambda^g \left(\int_{\underset{\substack{\uparrow \\ \text{Disconnected}}}{\overline{M}(g, \beta)}} C_{vir}(g, \beta) \right) q^{\beta} \qquad (8)$

$$q^{\beta} = e^{2\pi i \sum'_{\beta_i} \left(\int_{C_i} (iJ + B) \right) \cdot \beta_i} =$$

t_i ↑ Complexified Volumes of curves; β_i ↑ degrees

Customary to consider free energy

$$F = \log Z = \sum'^{\infty}_{\substack{g=0 \\ \beta \in H_2(M, \mathbb{Z})}} \lambda^{2g-2} \underset{\substack{\uparrow \\ \mathbb{Q}}}{\Gamma^{\beta}_g} q^{\beta}$$

$$\Gamma^{\beta}_g = \int_{\overline{M}_{g, \beta}} C_{vir}(g, \beta) \in \mathbb{Q}$$

Gromov - Witten invariants

Goal: calculate all r_g^β for given geometry $\rightsquigarrow F$ of A-model

Symplectic Invariants

$$Q = e^{it}$$

$$Z_{GV}(Q,q) = \prod_\beta \Big[\Big(\prod_{r=1}^{\infty} (1-Q^r q^\beta)^{r \cdot n_0^\beta} \Big) \times$$

$$\prod_{g=1}^{\infty} \prod_{l=0}^{2g-2} \left(1 - Q^{g-l-1} q^\beta\right)^{(-1)^{g+l} \binom{2g-2}{l} n_g^\beta} \Big] \quad (9)$$

$n_g^\beta \in \mathbb{Z}$ Gopakumar-Vafa invariants

Observation

ideal sheaf IDG D2D0 charge

$0 \to I \to \mathcal{O}_M \to \mathcal{O}_Z \to 0$

$$Z_{DT}(Q,q) = \sum_{\beta, k \in \mathbb{Z}} N_k^\beta Q^k q^\beta$$

D2D0 Brane BPS numbers

k: number of D0 Branes

$N_k^\beta \in \mathbb{Z}$ proven Donaldson-Thomas Invarianten

$$Z_{GV}(Q,q)\, M(Q)^{\frac{\chi(M)}{2}} = Z_{DT}(-Q,q)$$

$$M(Q) = \prod_{n \geq 1} \frac{1}{(1-q^n)^n} \quad \text{McMahon function} \quad (10)$$

generation function of 3-d partions

$$\dim_{\text{naiv}}(\overline{M}_{g,\beta}) = \int_\beta c_1(TM) + (g-1)(3 - \dim_{\mathbb{C}}(M)) \quad (11)$$

$= 0$ for CY-3fold $\forall\, g, \beta$

generically $r_g^\beta \neq 0$ counts "points"

Mathematical thechnique for calculating symplectic invariants is <u>toric</u> <u>localization</u>

Idea: $(\mathbb{C}^*)^d$ acts toric manifold B $\qquad \overbrace{K_B \to B}^{\text{toric CY}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \uparrow$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Toric Fano

Example $\quad \mathcal{O}(-3) \to \mathbb{P}^2$

$$\begin{tikzcd}
U_\beta = \pi_* ev^*(K_B) \arrow[d] & & ev^*(K_B) \arrow[ll] & K_B \arrow[l] \arrow[d] \\
(\Sigma, X) \in \overline{\mathcal{M}}_{g,0}(\beta, B) & & \overline{\mathcal{M}}_{g,1}(\beta, B) \arrow[ll, "\pi"'] \arrow[r, "ev"] & B
\end{tikzcd}$$

$$U_\beta = H^1(\Sigma, X^* K_B)$$

$$\Gamma_g^\beta = \int_{\overline{\mathcal{M}}_{g,0}(\beta, B)} c_{vir}(U_\beta) \qquad (11)$$

B -

- $\pi_* ev^*$ pulls $(\mathbb{C}^*)^d$ action back to $\overline{\mathcal{M}}_{g,0}(\beta, B)$
- (11) is calculated from combinatoric of fixpoints under $\pi_* e^*((\mathbb{C}^*)^r)$

<u>Toric manifolds</u>

$$\mathcal{M}_T = \left(\mathbb{C}^m \setminus Z(\{D_{i_1}, \ldots D_{i_s}\})\right) / (\mathbb{C}^*)^r$$

$$d = \dim_{\mathbb{C}}(\mathcal{M}_T) = m - r \qquad \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

Action of $(\mathbb{C}^*)^r$ is described on coordinates of $\mathbb{C}^m$ by

k-th torus action $\quad X_i \longmapsto \mu_{(k)}^{Q_i^{(k)}} X_i \qquad \mu_{(k)} \in \mathbb{C}^*$

$i = 1 \ldots m$

$k = 1 \ldots r$

$Q_i^{(k)}$ "charge" of i'th field under k,th

$G = \mathbb{C}^* \qquad G_{\mathbb{R}} = U(1)$

$Z(\{D\})$ Stanley Reisner ideal

$\mathbb{C}^m \setminus Z(\{D\})$ has smooth orbits under $(\mathbb{C}^*)^r$ & separable $G_{\mathbb{R}}$ Orbit

Example $\quad \mathbb{P}^2 \qquad Q^{(1)} = (1,1,1)$

$$Z(\{D\}) = \{X_1 = X_2 = X_3 = 0\}$$

$(\varkappa, \lambda)$ $(\mathbb{C}^*)^2$ acts on $\mathbb{P}^2$ $\quad (X_1, X_2, X_3) \longmapsto (\lambda \varkappa^{-1} X_1, \lambda^{-1} X_2, \varkappa X_3)$

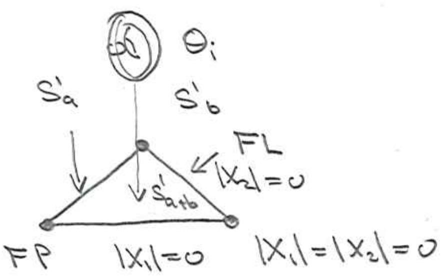
Fixpoints 3 $\quad X_i = X_k = 0 \qquad$ fix under $(\mathbb{C}^*)^2$

3 $\quad X_k = 0 \qquad$ fix under $\mathbb{C}^*$

Example of Localization

$\# FP = \int_{M_T} c_d(TM) = 3$

moment map



$X_i = |X_i| e^{i\Theta_k}$

Properties :- $\quad c_1(T_{M_T}) = 0 \iff \sum_{i=1}^{m} Q_i^{(k)} = 0 \quad \forall k$

- M_T symplectic form

$$\omega = \frac{i}{2} \sum_k^{1} dX_k \wedge d\bar{X}_k = \frac{1}{2} \sum^{1} d|X_k|^2 \wedge d\Theta_k$$

<u>Example</u> $\mathcal{O}(-3) \to \mathbb{P}^2$

$$\begin{array}{c} x_0 \\ Q^{(1)} = (-3, 1\,1\,1) \end{array}$$

$$Z(\{D\}) = \{x_1 = x_2 = x_3 = 0\}$$

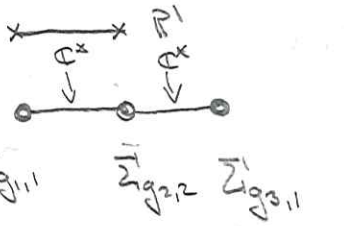
Fixpoints: $(\Sigma'_{g,h}, X) \xrightarrow{ev} M_T$
on $M_{g,0}(\beta, M_T)$

- 1) $\Sigma_{g,}$ $\to$ $FP \in M_T$
- 2) $\Sigma'_{g,}$ $\to$ $FL \in M_T$

only possible if $\Sigma_g = \mathbb{P}^1$

or $\Sigma'_g = \mathbb{C}^*$ 

Only degenerate maps contribute



$\leadsto$ evaluate $\int_{M_{\alpha,\beta}} c_{top}(U_\beta)$ Kontsevich 94 Quintic $g=0$
Klemm Zaslow 99 toric CY $\forall g$
based on Faber & Pandharipande 97

<u>The topological Vertex</u> hep-th/0305132

Aganagic, Mariño, Vafa, AK

Consider toric CY - 3 fold M_T

$Q_i^{(k)}$ $\quad i = 1 \dots r+3$, $k = 1 \dots r$ $\quad \sum_{i=1}^{r+3} Q_i^{(k)} = 0$

$Z(\{D\})$

- M_T covered by $\mathbb{C}^3$ patches
- M_T admits Harvey Lawson special Lagrangians

$\mathbb{C}^3$ patch $\quad (x_1, x_2, x_3) \in \mathbb{C}^*$

a) T^3 fibration $\quad$ b) $T^2 \times \mathbb{R}$ fibration

$\uparrow$ parametrized by Θ_i $\quad$ $\uparrow$ parametrized by α_1, α_2, r_R

Symplectic form $\quad \omega = \frac{1}{2} \sum_i^{'} d|x_i|^2 \wedge d\theta_i$

Hamiltonians

$$r_{\alpha_1} = |x_1|^2 - |x_2|^2$$

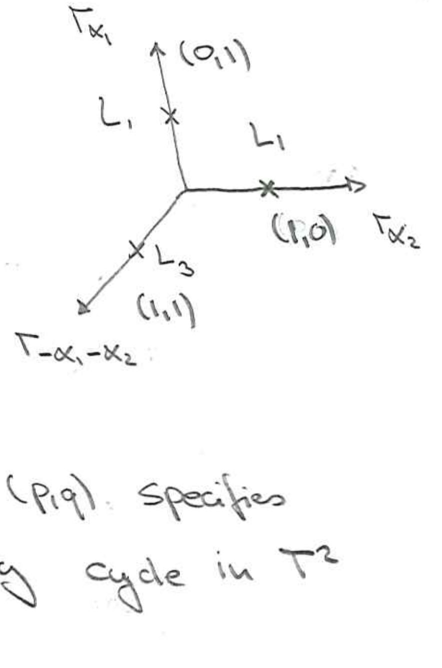
$$r_{\alpha_2} = |x_3|^2 - |x_1|^2$$

$$r_{\mathbb{R}} = \operatorname{Im}(x_1 x_2 x_3)$$

Hamitonian flow $\quad \partial_\alpha x_k = \{ r_{\alpha_1}, x_k \}$

$$e^{\alpha_1 r_{\alpha_1} + \alpha_2 r_{\alpha_2}} : (x_1, x_2, x_3) \longmapsto (e^{i(\alpha_2 - \alpha_1)} x_1, e^{-i\alpha_1} x_2, e^{i\alpha_2} x_2)$$

L_1 : $\quad r_{\alpha_1} = 0 \quad \underset{(*)}{r_{\alpha_1}} = \underset{(2)}{s_1} \quad r_R \geq 0 \quad \underset{(**)}{\operatorname{Re}(x_1 x_2 x_3) = 0}$



$$L_1 \sim \mathbb{C} \times S^1$$

Lagrangian: $\quad \omega|_L = 0$

$(*)$ $\quad d|x_1|^2 = d|x_2|^2 = |dx_3|^2 = 0$

$(**)$ $\quad d(\Theta_1 + \Theta_2 + \Theta_3) = 0$

$$\omega|_L = \frac{1}{2} |dx_3|^2 \wedge d(\theta_1 + \theta_2 + \theta_3) = 0$$

Slope (p,q) specifies vanishing cycle in T^2

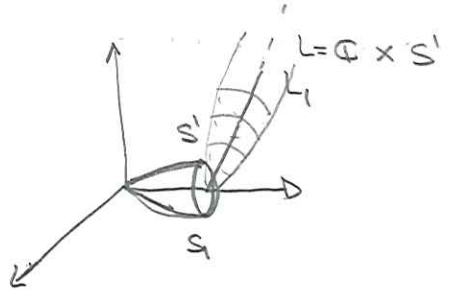
Special Lagrangian

$$\Omega|_L = e^{i\alpha} dr_R \wedge d\alpha_1 \wedge d\alpha_2 = e^{i\alpha} \operatorname{vol}(L)$$

L_2 : $\quad r_{\alpha_1} = + s_2 \quad r_{\alpha_2} = 0 \quad$ " $\quad$ "

L_3 $\quad r_{\alpha_1} + r_{\alpha_2} = 0 \quad r_{\alpha_1} = - s_3 \quad$ " $\quad$ "

3d

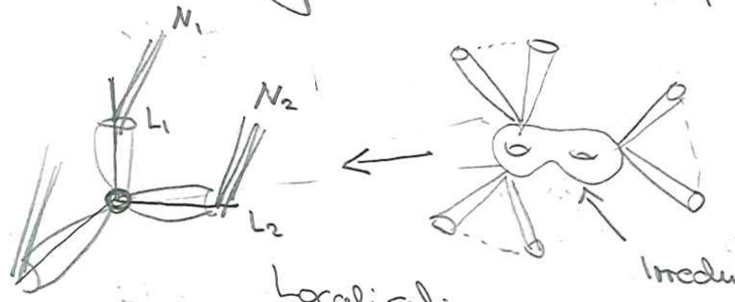


open string parameter

$$u_\alpha = i s_\alpha + \oint_{S^1_\alpha} A_\alpha$$

Consider $\Sigma_{g,h}$ which is mapped

to geometry $(\mathbb{C}^3 \ (L_1^{N_1}, L_2^{N_2} \ L_3^{N_3})$



Configuration specified by g, winding #

$k_j^{(\alpha)}$ $\quad \alpha = 1,2,3$

$\|$

# of holes ending on α's brane

with winding j

and framing f_α $\quad \alpha = 1,2,3$

Winding basis $\quad Z = \sum_{\vec{k}^{(\alpha)} \ \alpha=1,2,3} C^{(f_1 f_2 f_3)}_{\vec{k}^{(1)} \vec{k}^{(2)} \vec{k}^{(3)}}(Q) \prod_{\alpha=1}^{3} \frac{1}{Z_{\vec{k}^{(\alpha)}}} \mathrm{Tr}_{\vec{k}^{(\alpha)}} V_\alpha$

exact in string coupling

$$V_\alpha = P\left[\oint A_\alpha\right]$$

$$Z_{\vec{k}} = \prod_j k_j! \, j^{k_j}$$ Order of Automorphism group

Representation Basis.

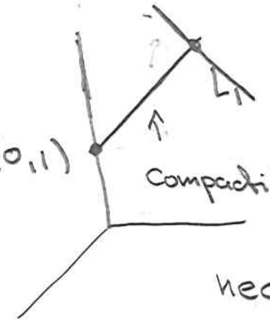
Basis trans between symmetric functions

$$\mathrm{Tr}_{\vec{k}} V = \underbrace{\sum_R \chi_R(C(\vec{k}))}_{} \mathrm{Tr}_R V$$

character of symmetric Group

Framing ambiguity: Open Amplitudes depend on the boundary conditions of L at infinity

$V = (0,1)$



$f_1 = (p_1 q_1)$

compactify L to S^3

necessary

$$V_1 f_2 - V_2 f_1 = V \wedge f = 1$$

$$f \mapsto f - nV \qquad n \in \mathbb{Z}$$

framing ambiguity

Canonical framing $V_i = f_i$

$$C_{R_1 R_2 R_3}^{(f_\alpha - n_\alpha v_\alpha)} = (-1)^{\sum_i' n_\alpha \, l(R_\alpha)} \, Q^{\sum_i' n_\alpha \frac{\varkappa_{R_\alpha}}{2}} \, C_{R_1 R_2 R_3}^{(f_\alpha)}$$

$C_{R_1 R_2 R_3}^{(f_1 f_2 f_3)}$ cyclic symmetric

R

$l(R) = \#$ of boxes

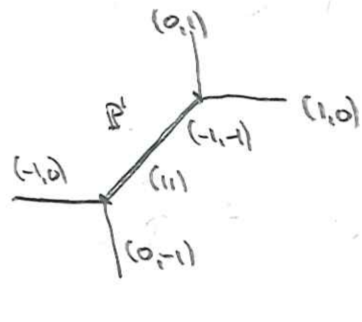
$$\varkappa(R) = l(R) + \sum_k' (l_k^2 - 2k\, l_k)$$

Note: – In localization n is an dependence of amplitudes on choice of torus action math AG 0103074

– In Chern-Simons n corresponds to framing of Links hep-th 0105045

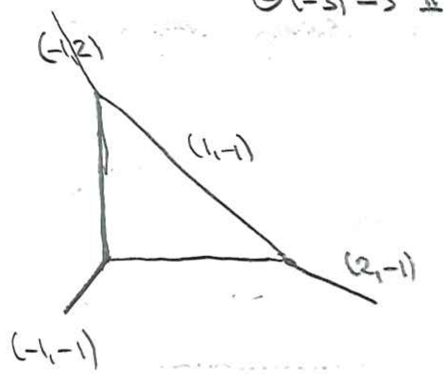
Gluing: M_T can be build from $\mathbb{C}^3$ patches

E.g.

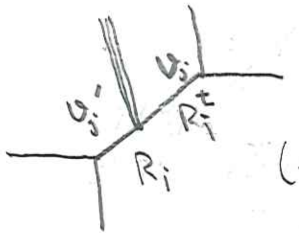


$$\mathcal{O}(-1) \oplus \mathcal{O}(-1)$$
$$\downarrow$$
$$\mathbb{P}^1$$

$\mathcal{O}(-3) \to \mathbb{P}^2$



Complete set of boundary condition



$(-1)^{\ell(R_i)} e^{-\ell_i(R_i)t} \delta_{R_1 R_2^t}$

$v_j' \wedge v_j = n_i$

Kähler of $\mathbb{P}^1$

$\Downarrow$

$$\sum_{R_i}' C_{R_j R_k R_i} e^{-\ell(R_i)t} (-1)^{(n_i+1)\ell(R_i)} Q^{-n_i \frac{\varkappa_{R_i}}{2}} C_{R_i^t R_j' R_k'}$$

Explicitly:

$$C_{R_1R_2R_3}(Q) = \sum_{R,S,S_2}' \underset{\substack{\uparrow \\ \text{tensor product coeff}}}{N^{R_1}_{S_1 R}} N^{R_3^t}_{Q_3^t R} Q^{\frac{\varkappa_{R_2}+\varkappa_{R_3}}{2}} \frac{W_{R_2^t S_1} W_{R_2 S_2^t}}{W_{R_2}}$$

$\uparrow$ Hopf Link invariants

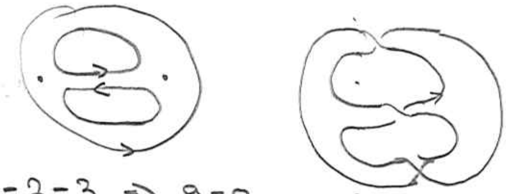
<u>Chern-Simons theory</u>

$$S_{CS} = \frac{2\pi}{k+N} \int_{M_3} \mathrm{Tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A\right)$$

A $U(N)$ connection $\quad \frac{2\pi}{k+N} \sim \frac{1}{g_{CS}}$ coupling

Perturbatively one has fat graph expansion

$2g-2 = E - V - h$



Example $\quad 3-2-3 \Rightarrow g=0 \qquad 3-2-3 \Rightarrow g=1$

$$F = \sum_{g,h}' F_{g,h}\, g_s^{2g-2}\, t^h \qquad t = N \cdot g_s$$

If it is possible to sum over $h \rightsquigarrow$

$$F = \sum_{g}' F_g(t)\, g_s^{2g-2}$$

closed string expansion

Chern-Simons theory & open String theory

Consider topological String on $T^* M_3$ cotangentical bundle to S^3. Noncompacta CY in which M_3 is Lagrangian.

$$\omega = \sum_{k=1}^{3} dp_k \wedge dq_k$$

Fibre section cover

Possible corrections are open holomorphic Instantons

$$X : \Sigma_{g,h} \to T^* M_3$$

$$X(\partial \Sigma) \subset M \qquad d\varsigma = \omega \quad \text{with}$$

Instanton action

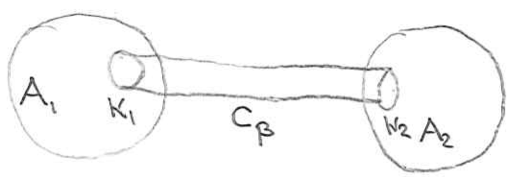
$$\varsigma = \sum_{k=1}^{3} p_k \, dq_k \qquad \int_{\Sigma} X^*(\omega) = \int_{\partial \Sigma} X^*(\varsigma) = 0$$

$$\varsigma|_{M_3} = 0$$

$\Rightarrow$ only degenerate Instantons, which correspond to fatgraphs

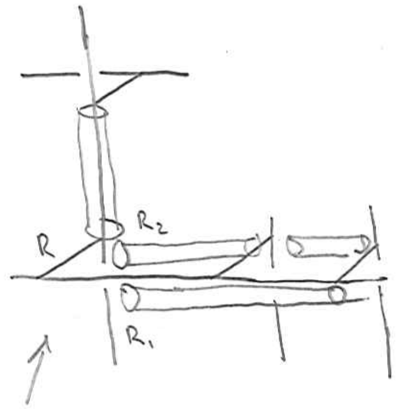
Consider CY with several conifolds: Shrinking $S^3 \rightsquigarrow$ locally CY looks like $T^* S^3$

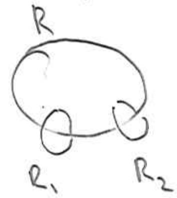
$$S(A_i) = \sum_i S_{CS}(A_i) + \sum_\beta e^{-\int_{C_\beta} \omega} \prod_i \mathrm{Tr}\, U_{K_i(\beta)}$$



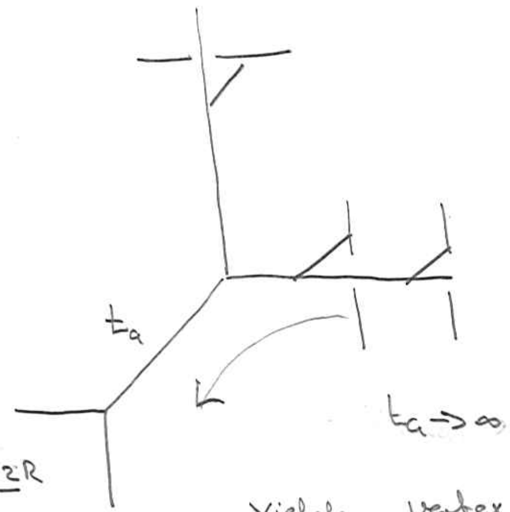
$$U_{K_i} = P e^{-\int_{K_i} A_i}$$

Vertex konfiguration





$$W_{R_1 R_2 R} = \frac{W_{R_1 R}\, W_{R_2 R}}{W_R}$$



Yields vertex in particular framing Link inv.

$\Rightarrow$ we get formula in terms of alternatively Okounkov Reshetikhin Vafa

$$C_{R_1 R_2 R_3} = Q^{\frac{1}{2}(\kappa_{R_2} + \kappa_{R_3})} \, S_{R_1^t}(Q^{\varsigma}) \sum_{R}{}' S_{R_1/R}\left(Q^{\ell(R_2^t) + \varsigma}\right) S_{R_3^t/R}\left(Q^{\ell(R_2) + \varsigma}\right)$$

$\varsigma = (-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$

$S_{R/P}$ relative Schur functions

Example

$$C_{\substack{\square\\\square}\,\square\,\square} = \frac{Q^6 - Q^5 + Q^3 - Q + 1}{Q(Q-1)^3 (Q^2-1)}$$

Relation to integrable Systems

Mirror symmetry: Comp. Kähler Str. $\downarrow$ complex structure $\downarrow$

$$Z(M, t, u) = Z(W, \hat{t}, \hat{u})$$

mirror manifolds

$$H^{p,q}(W) = H^{\dim - p, q}$$

near Maximal monodromy point

$$u = \hat{u} + \frac{t - \hat{t}}{3}$$

$$t = \log(\hat{t}) + \sum' a_n e^{-\hat{t}}$$

Period integral

Example: MS for non-compact toric CY

Data: $Q_i^{(\alpha)}$ charges of GLSM $i = 1, \dots, r+d$

Mirror geometry, Katz, Klemm Vafa, Hori & Vafa

$$H(Y) := \sum_{i=1}^{r+d} Y_i = 0 \qquad Y_i \in \mathbb{C}^*$$

$$\vec{Y} \sim \lambda \vec{Y} \qquad \lambda \in \mathbb{C}^*$$

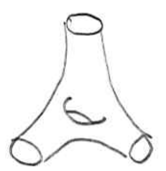
$$\prod_{i=1}^{r+d} Y_i^{Q_i^{(\alpha)}} = e^{\hat{t}^{(\alpha)}}$$

Example $\mathcal{O}(-3) \to \mathbb{P}^2$

$$Y_0 + \underset{X}{\underset{\parallel}{Y_1}} + \underset{Y}{\underset{\parallel}{Y_2}} + Y_3 = 0$$

$$H(X, Y) = 1 + X + Y + \frac{e^{\hat{t}}}{XY} = 0$$

$\leftarrow$ complex structure of $\mathcal{C}_{g=1}$



Mirror geometry W: $H(X, Y) = Z \cdot W$

non-compact 3-fold

Variation of CS of W encoded in

$\mathcal{C}_g$ $H(X, Y) = 0$ more precisely

the periods of $\lambda = \Omega|_{\mathcal{C}_g} = \log(X)\, d\log(Y)$

Conjecture. Mariño, Bouchard, Mariño, Pasquetti A.K

$H(X,Y)=0$ can be viewed as spectral curve of a Matrix model with

$x \sim\sim x$ $\quad x \sim\sim x$ $\quad \int_{x_i}^{x_{i+1}} \lambda$ defining the filling fractions

$x_i \quad x_{i+1} \quad \ldots\ldots \quad x_i$

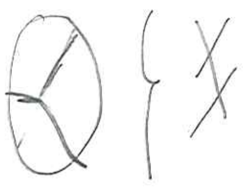
Evidence: Correlation function calculated from recursive evaluation of Loop-equation

....., Eynard & Orantin 07 + mirror map at MMP

$\rightsquigarrow$ topological vertex result

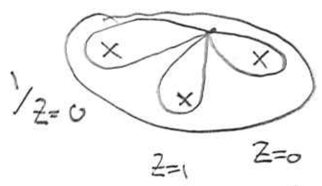
Extension: Consider $C_g(\hat{t})$ as complex family of curves.

$\mathcal{O}(-3) \to \mathbb{P}^2$ example



$$\Pi = \begin{pmatrix} \int_a \lambda \\ \int_b \lambda \\ \int_\gamma \lambda \end{pmatrix}$$

$\uparrow$ around residue



$e^{\hat{t}} = z$

$\mathbb{Z}_3$ Orbifold Conifold MMP.

$\Gamma_0(3)$

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 3 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 1 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \qquad \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ -1 & 1 \end{matrix}} & \begin{matrix} 3 \\ 1 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

$$F_g = \underset{\substack{\uparrow \\ \text{weight}-3}}{X^{g-1}} \sum_{k=0}^{3g-3} \hat{E}_2^k \underset{\substack{\uparrow \\ \text{weight } 6g-6-k}}{c_k(d,g)}$$