Higher Spin Gauge Theory & Holography

Part I Motivations and Overview.

- Why higher spin?

Flat Minkowskian space – no non-trivial S-matrix of massless fields of spin $S>2$. [Weinberg ← assumes min. coupling, Coleman - Mandula, Aragone, Deser]

This is no longer the case in curved space. (AdS, dS, for instance)

$$\varphi_{\mu_1\mu_2\cdots\mu_s}(x)$$

← symmetric, traceless tensor.

[in fact, to write gauge invt. e.o.m, need double-traceless tensor fields.
e.g. $S=2$, graviton $h_{\mu\nu}$. generally not traceless, trace part can be gauged away]

gauge sym takes the form

$$\delta\varphi_{\mu_1\cdots\mu_s}(x) = \partial_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)}(x) + \cdots$$

$\epsilon_{\mu_1\cdots\mu_{s-1}}$ traceless tensor gauge parameter.

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String theory has a tower of massive fields with mixed symmetry wrt Lorentz group.

Leading Regge trajectory $\longrightarrow$ symmetric traceless tensor.

$$\underbrace{\boxed{\ }\boxed{\ }\cdots\boxed{\ }\boxed{\ }}_{s}$$

High $\cdot$ E limit

$\rightarrow$ massless HS fields ?

Not entirely clear in flat space [Attempts by Sagnotti, et al.]

But realized in AdS (in tensionless limit

$$\alpha' = l_s^2 \gg R_{AdS}^2 .$$)

Our most familiar example

IIB on $AdS_5 \times S^5$ $\longleftrightarrow$ 4d $\mathcal{N}=4$ SYM.

$U(N)$ gauge group. $\lambda = g_{YM}^2 N$.

$$g_s \longleftrightarrow g_{YM}^2 = \frac{\lambda}{N}.$$

$$\frac{R}{l_s} \longleftrightarrow \lambda^{\frac{1}{4}}.$$

gauge theory : two expansion parameters.

$$\frac{1}{N} \text{ and } \lambda .$$

For FINITE λ, $\frac{1}{N} \sim g_s$ in bulk.

$N \gg 1 \Rightarrow$ weakly coupled String Field Theory

even when R/l_s is not large.

$\lambda \to \infty$ limit $\Rightarrow$ supergravity on $AdS_5 \times S^5$.

$\lambda \to 0$ limit $\Rightarrow$ higher spin gauge theory (HSGT)
+ (massive) matter on AdS_5.

take Φ to be one of the 6 scalars in adjoint of $U(N)$

Have HS currents, of the form

$$J^{(s)}_{\mu_1 \cdots \mu_s} = \mathrm{Tr}\left(\Phi \overleftrightarrow{D}_{(\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_s)} \Phi\right) + \cdots$$

symmetric, traceless. subtract off traces, etc.

$\lambda \to 0$ limit, $J^{(s)}_{\mu_1 \cdots \mu_s}$ conserved.

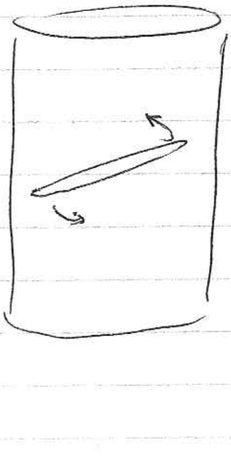
$$\partial^{\mu_1} J^{(s)}_{\mu_1 \mu_2 \cdots \mu_s} = 0. \quad \Rightarrow \text{HS symmetry.}$$

[$\lambda > 0$, $J^{(s)}$ acquires anomalous dimension.

$$\Delta = s + 2 + \delta_s.$$

$$\delta_s = f(\lambda) \ln s + \cdots, \quad s \gg 1$$

cusp anomalous dim



$\lambda \gg 1$ matches $\Delta - s$ of semi-classical spinning folded string in AdS.

will return to this later]

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HS sym of the boundary CFT

$\updownarrow$

HS gauge field in the bulk,

$\varphi^{(s)}_{\mu_1 \cdots \mu_s}$ coupled to $J^{(s)}_{i_1 \cdots i_s}$ on bdry.

mass of $\varphi^{(s)}$ fixed by gauge sym. $\leftarrow$ [may call them "massless" in AdS]

$\longleftrightarrow$ $\Delta = s+1$ for dual current $J^{(s)}$.

Also have lots more in the bulk,

elementary bulk fields $\longleftrightarrow$ single-trace operators in bdry CFT.

e.g. $\mathrm{Tr}(\Phi^n)$, $n = 2, 3, \cdots$

[cut off @ N]

$\updownarrow$

a tower of massive bulk fields, $s=0$.

So, $\mathcal{N}=4$ SYM @ $\lambda \to 0$ (gauged at zero coupling)

is dual to a HSGT + many massive fields in AdS_5.

$G_N \sim \frac{1}{N^2}$.

This HSGT + matter is the tensionless limit of IIB string field theory in $AdS_5 \times S^5$.

Q: What is the simplest HSGT?

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- HSGT can be realized simply in AdS_3 via Chern-Simons, coupling to matter fields tricky, but possible.
  - 3d gravity has no bulk d.o.f., too trivial for some purposes (may still learn interesting aspects of black holes)

- <u>4-dim</u> Vasiliev '92.

  A <u>pure</u> HSGT in AdS_4.

  minimal version: contains 1 gauge field of each even ≥ 0 spin.

$$S = \underset{\substack{\text{a scalar} \\ \text{w/ } m^2 = -\frac{2}{R_{AdS}^2}}}{0}, \underset{\text{graviton}}{2}, \underbrace{4, 6, \ldots\ldots}_{\text{HS fields}}$$

  - Need ∞ tower! (just like string field th.)

  "non-minimal" version; $S = 0, 1, 2, 3, 4, \ldots$

  all ≥ 0 integer spins included.

  [• can also include fermions, SUSY, etc.]

Interactions <u>highly constrained</u>

Almost uniquely fixed by HS gauge symmetry!

If demand parity

only two theories, fixed up to overall coupling.

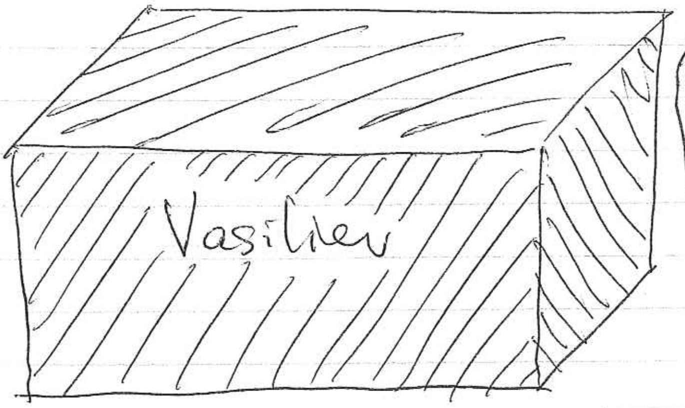
A-type: bulk scalar $\varphi^{(0)}$, parity even

B-type: $\varphi^{(0)}$ parity odd.

If relax parity, $\exists$ ∞-parameter family of (minimal or non-minimal bosonic e.g.) Vasiliev theory.

interactions still very constrained.

- one new coupling parameter for $(n+3)$-th order interaction, $n = 0, 2, 4, \ldots$



We will unlock this black box ~~in the next lecture~~ SOON!

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Some features of Vasiliev's theory.

- Know full nonlinear classical eqn of motion.
- but, in 1^{st} order formalism, w/ many aux. fields.
- linearized eqn $\Rightarrow$ Fronsdal eqn, standard free HS gauge fields.
- There appears to be no obstruction in recovering full Lagrangian order by order in power of fields from the e.o.m. (by solving aux. fields).

  in practice very hard, has not been done.
- A priori, arbitrarily high # deriv. interaction at each order in HS fields,
  - @ cubic level, finitely many independent <u>on-shell</u> vertices for given triple of spins (s_1, s_2, s_3).

e.g. scalar φ, $\lambda\varphi^3$ coupling (actually, $\lambda = 0$, in AdS_4).

What about $\varphi\,\partial^\mu\varphi\,\partial_\mu\varphi$?

$$\varphi\,\partial^\mu\varphi\,\partial_\mu\varphi = \underbrace{\tfrac{1}{2}\partial_\mu(\varphi^2\partial^\mu\varphi)}_{\text{total deriv.}} - \tfrac{1}{2}\varphi^2\underbrace{\Box\varphi}_{= m\varphi + \mathcal{O}(\varphi^2) \text{ by e.o.m.}}$$

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- quartic & higher order interaction vertex.
  - ∞-ly many higher deriv. terms.
  - presumably non-local.

  (analogous to SFT, $\frac{l_s}{R} \to \infty$ limit)

- Interactions almost entirely fixed by HS sym.
  If HS sym preserved by quantum corrections,
  can only renormalize overall coupling constant.
  - conceivably a <u>UV finite</u> theory,
    despite ∞-ly many higher deriv. terms.
  - quantum corrections have not been studied explicitly
    (gauge fixing, regularization ......)

Now, if Vasiliev's system describe a consistent quantum theory of gravity + HS fields in AdS_4

it must have a CFT dual,

with some sort of $1/N$ expansion,

$$g_{BULK} \sim \frac{1}{N}.$$

What is this dual CFT?

Zhiboedov & Maldacena '11.

EXACT HS sym $\Rightarrow$ CFT$_3$ is free

★ does NOT mean that there is a free massless scalar or fermion field.

On the other hand, HS sym can be broken by AdS boundary condition.

$\Rightarrow$ more interesting dual CFTs.

Klebanov, Polyakov '02.
Sezgin, Sundell
Giombi, Minwalla, Prakash, Trivedi, Wadia, XY, '11.

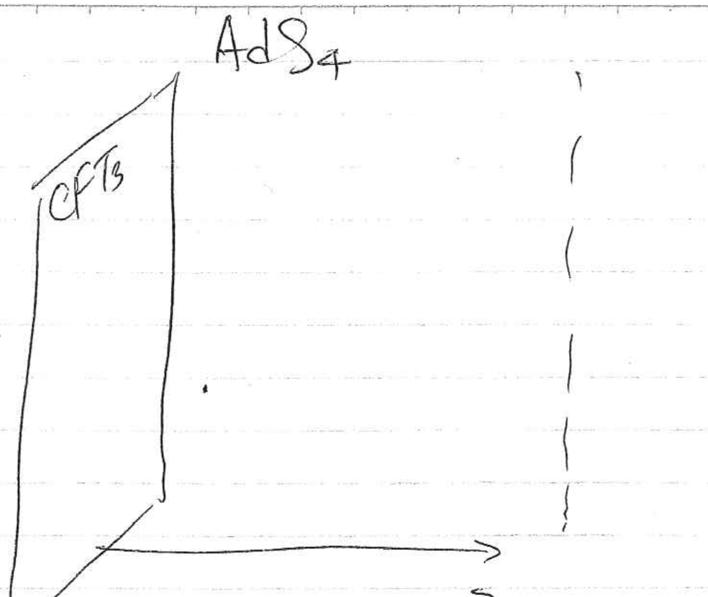
Dual CFTs are "vector models"

★ Large classes of critical (2+1)-D systems in condensed matter physics are such vector models (w/ small or "modest" values of N)

and we (conjecturally) know their EXACT holographic dual

[Note: NO SUSY, though admit SUSY generalization]

To list a few:
critical O(N) model,
QED + scalar or fermion flavors,
$\mathbb{CP}^{N-1}$ model, ...



can impose 2 different bdry conditions on bulk scalar φ.

$$m^2 = -\frac{2}{R^2}.$$

$$\updownarrow$$

dual to scalar operator $\mathcal{O}$ in CFT_3

with scaling dimension

$$\Delta = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 + (mR)^2} = \begin{cases} 2 \\ 1 \end{cases}.$$

$$ds^2 = \frac{d\vec{x}^{\,2} + dz^2}{z^2}. \qquad \vec{x} \in \mathbb{R}^3 \text{ or } \mathbb{R}^{1,2}$$

"$\Delta = 2$ bdry condition"

$$\varphi(\vec{x}, z) \sim z^2 + \mathcal{O}(z^3)$$ in the absence of source.

or

$$z + \mathcal{O}(z^3). \qquad z \to 0.$$

"$\Delta = 1$ bdry condition"

[for HS fields $\varphi_{\mu_1 \cdots \mu_s}$, can only impose one bdry condition

$$\varphi_{i_1 \cdots i_s}(\vec{x}, z) \sim z. \quad (z \to 0)$$

(sources gives z^{2-2s})]

Minimal Vasiliev theory.

	A-type	B-type
$\Delta=1$ b.c.	free O(N) theory.	critical O(N) Gross-Neveu model
$\Delta=2$ b.c.	critical O(N) model.	free O(N) fermions.

explain this first.

free O(N) theory. (in 3d.) take N massless real free scalars ϕ_i

restrict to O(N) - singlet sector.

i.e. operators of this CFT are O(N)-singlets made out of ϕ_i & their derivatives.

$$\phi_i\phi_i, \quad \phi_i\partial_\mu\phi_i, \quad \phi_i\partial_\mu\partial_\nu\phi_i, \quad \phi_i\phi_i\phi_j\phi_j, \cdots$$

OPEs close $\Rightarrow$ a well-def'd CFT.

- Note that the O(N)-singlet condition is what allows for a $1/N$ expansion. Without this condition there cannot be a weakly coupled holographic dual

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- Spectrum of single trace operators.

for simplicity let us consider "free U(N) theory" for the moment, ϕ_i cplx scalars.

$$\partial\cdots\partial\bar{\phi}_i\ \partial\cdots\partial\phi_i ,$$

organize into rep of conformal algebra $SO(3,2)$ or $SO(4,1)$ (Euclidean).

single "letter" character count $\partial\cdots\partial\phi$'s

e.o.m.

$$\mathrm{Tr}_\phi\ x^\Delta \mu^{J_3} = \frac{x^{\frac{1}{2}}(1-x^2)}{(1-x)(1-\mu x)(1-\mu^{-1}x)}.$$

character of all single trace op's

$$\frac{x(1+x)^2}{(1-\mu x)^2(1-\mu^{-1}x)^2} \xrightarrow{\mu=1} \frac{x(1+x)^2}{(1-x)^4},$$

primary operators labeled by (Δ, j)

spin.

$$(\Delta, j) = \underset{\text{long rep.}}{(1,0)},\ \underbrace{(2,1),\ (3,2),\ \ldots,\ (s+1,s)}_{\text{short rep}}, \ldots$$

$$G_{\Delta,s} = \frac{(2s+1)x^\Delta}{(1-x)^3}.$$ char of long rep.

$$\chi_{s+1,s} = G_{s+1,s} - G_{s+2,s-1}$$ char of short rep.

subtract off null states $\partial^\mu J^{(s)}_{\mu\ldots}$

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$$\frac{x(1+x)^2}{(1-x)^4} = G_{1,0} + \sum_{S=1}^{\infty} \chi_{S+1,S} .$$

precisely one copy of $(S+1, S)$ for each $S=0,1,2,\cdots$.

$$J^{(0)} = \bar{\phi}_i \phi_i ,$$

$$J^{(1)}_{\mu} = \bar{\phi}_i \overleftrightarrow{\partial}_{\mu} \phi_i ,$$

$$J^{(2)}_{\mu\nu} = \bar{\phi}_i \overleftrightarrow{\partial}_{\mu} \overleftrightarrow{\partial}_{\nu} \phi_i - 2\partial_{(\mu} \bar{\phi}_i \partial_{\nu)} \phi_i + 2\delta_{\mu\nu} \partial^{\rho} \bar{\phi}_i \partial_{\rho} \phi_i ,$$

$$J^{(3)}_{\mu\nu\rho} = \cdots\cdot$$

etc.

for $O(N)$, ϕ_i real, only keep $J^{(s)}$, $s=$ even.

These of course match the (free) spectrum of minimal A-type Vasiliev system in AdS_4.

$\mathcal{O} = \bar{\phi}_i \phi_i$ is parity even

$\Delta = 1 \Rightarrow \Delta = 1$ bdry condition.

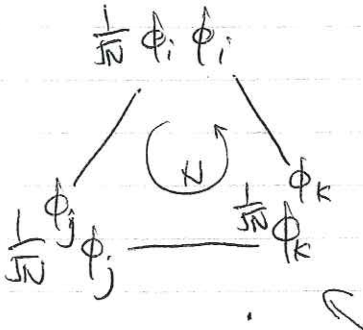
What about interactions?

correlation function

normalize two-point fn $\langle \mathcal{O}\mathcal{O} \rangle \sim N^0$,

$$\mathcal{O} = \frac{1}{\sqrt{N}} \phi_i \phi_i$$

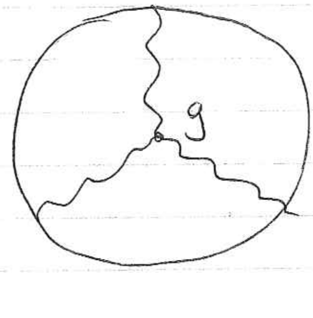
Now, 3-pt fn. $\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2)\, \mathcal{O}(x_3) \rangle$



$\sim N^{-\frac{1}{2}}$

$$= N^{-\frac{1}{2}} \cdot \frac{1}{|x_{12}|\,|x_{23}|\,|x_{31}|}$$

bulk Witten diagram.



$g_{BULK} \sim N^{-\frac{1}{2}}$.

subtlety: actually, ϕ^3 coupling vanishes.

$$\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle_{\substack{\Delta=1 \\ \text{b.c.}}} = 0 \times \infty$$

ϕ^3 coupling $\quad \int_{AdS_4} K_{\Delta=1} K_{\Delta=1} K_{\Delta=1}$

need to be regularized.

We will see that all 3-pt fns at tree level $\langle J^{(s_1)} J^{(s_2)} J^{(s_3)} \rangle$ in Vasiliev th. w/ $\Delta=1$ b.c. match that of free $O(N)$ th., including relative normalization [Giombi, XY, '09, '10].

- After the work of Zhiboedov + Maldacena, this is viewed as a check that Vasiliev's th. makes sense perturbatively and that HS sym is preserved by $\Delta=1$ bdry condition $\leftarrow$ (this depends on bulk interaction!)

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☆ Critical $O(N)$ model.

$\vec{\phi} = (\phi_1, \cdots, \phi_N)$ massless real scalars.

$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 + \cdots$

$\mathcal{O} = \vec{\phi}^2$ free $O(N)$ vector model.

$\Delta = 1$

$\mathcal{O}^2 \Big\downarrow$ double trace def^n $(\phi_i \phi_i)^2$.

RG flow

α Critical $O(N)$ vector model

$\Delta = 2 + \mathcal{O}(\frac{1}{N})$

$\alpha \Big\downarrow$ NLσM on S^{N-1}

free $O(N-1)$ vector model.

NLσM description of the critical point:

$$\int D\vec{\phi}\, D\alpha \; e^{-N\int d^3x \left[\frac{1}{2}(\partial_\mu \vec{\phi})^2 + \frac{1}{2}\alpha(\vec{\phi}^2 - \frac{1}{g})\right]}.$$

$$= \int D\alpha \; e^{-\frac{N}{2}\underbrace{\left[\mathrm{Tr}\ln(-\Delta+\alpha) - \frac{1}{g}\int\alpha\right]}_{S_{eff}[\alpha]}}.$$

Large N, saddle pt $\langle\alpha\rangle = m^2$ obeys.

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{p^2+m^2} = \frac{1}{g}. \quad \text{(def'd w/ dim reg)}$$

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$$\Rightarrow \quad m^2 = \left[(4\pi)^{-\frac{d}{2}} \Gamma(1-\tfrac{d}{2})\, g \right]^{-\frac{2}{d-2}}.$$

critical pt

$m \to 0, \quad g \to \infty \qquad (2 < d < 4).$

expand $S_{eff}[\alpha]$ to quadratic order,

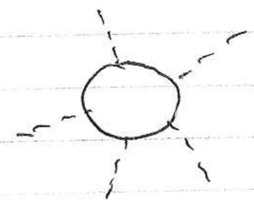
get non-local kinetic term for α.

(generated from $\int$ out $\vec{\phi}$ @ 1-loop).

$\langle \alpha(x)\, \alpha(0) \rangle \sim \frac{1}{(x^2)^2}. \quad (\times \frac{1}{N}).$

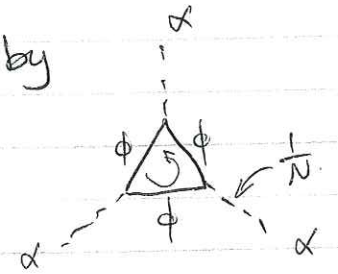
↑ a dim 2 operator. (at large N)

α^n vertex (also non-local)



e.g. $\langle \alpha\alpha\alpha \rangle$ computed to leading order in $1/N$

by



$\langle \alpha\alpha \rangle \sim \frac{1}{N}$

$\langle \alpha\alpha\alpha \rangle \sim N \times \left(\frac{1}{N}\right)^3 = \frac{1}{N^2}.$

$\sqrt{N}\alpha$ has canonically normalized 2-pt fn

and 3-pt fn $\sim \frac{1}{\sqrt{N}}$.

Actually, $\langle \alpha(x_1)\, \alpha(x_2)\, \alpha(x_3) \rangle$ vanishes at leading order

in $1/N$ - it is a contact term.

This is consistent w/ coeff. of φ^3

vanishes in bulk theory.

$\Delta=1$, $\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle = 0 \times \infty = \text{finite}.$

$$\uparrow \int_{AdS_4} K^3_{\Delta=1}$$

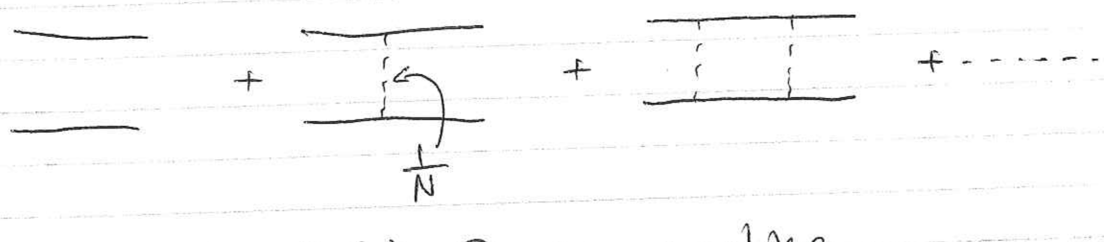
$\Delta=2$, $\langle \alpha\,\alpha\,\alpha \rangle = 0 \times \text{finite} = 0.$

$$\uparrow \int_{AdS_4} K^3_{\Delta=2}$$

[but other correlators nontrivial at this order.]

Can do a systematic $1/N$ - expansion

e.g. $\langle \phi_i\, \phi_j\, \phi_k\, \phi_\ell \rangle$.



Now gauge $O(N)$ @ zero coupling.

i.e. restrict to $O(N)$ - singlet operators

α, $\Delta=2$, dual to SAME bulk scalar

as in case of free $O(N)$ th.

but now w/ $\Delta=2$ bdry. condition.

Introducing Vasiliev's system. (4-dim)

x^μ - 4d spacetime coordinates

$y_\alpha, \bar{y}_{\dot\alpha}, z_\alpha, \bar{z}_{\dot\alpha}$, $\alpha = 1, 2$, $\dot\alpha = \dot 1, \dot 2$ twistor variables. (~~[illegible]~~ commuting spinors).

Define a $*$-algebra (non-commutative) on functions of (y, z) and $(\bar y, \bar z)$ separately.

$$f(y,z) * g(y,z) = f(y,z)\, e^{\epsilon^{\alpha\beta}\left(\frac{\overleftarrow{\partial}}{\partial y^\alpha} + \frac{\overleftarrow{\partial}}{\partial z^\alpha}\right)\left(\frac{\overrightarrow{\partial}}{\partial y^\beta} - \frac{\overrightarrow{\partial}}{\partial z^\beta}\right)} g(y,z)$$

$$= \int \underline{d^2u\, d^2v}\; e^{u \cdot v}\, f(y+u, z+u)\, g(y+v, z-v)$$

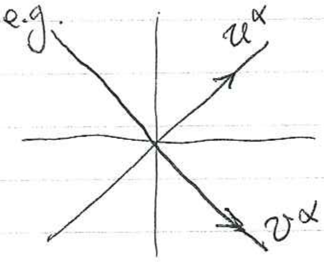
$u^\alpha v_\alpha = u^\alpha v^\beta \epsilon_{\beta\alpha}$.

Contour chosen so that $u \cdot v \in i\mathbb{R}$.

measure normalized. So that

$1 * f = f * 1 = f$.

e.g.



- $*$-product well defined on a set of "nice" functions.
- will encounter subtleties in doing pert. th

choice of contour is subtle.

"master fields"

1-form $\mathcal{A}(x \mid \overset{(y_\alpha, \bar{y}_{\dot\alpha})}{Y}, \overset{(z_\alpha, \bar{z}_{\dot\alpha})}{Z})$ on $(x^\mu, z^\alpha, \bar{z}^{\dot\alpha})$-space

i.e. involve only dx^μ, dz^α, $d\bar{z}^{\dot\alpha}$.

scalar $\mathcal{B}(x \mid Y, Z)$. (no dy's).

They will both contain HS fields and ∞-ly many auxiliary fields, related by e.o.m.

write $\mathcal{A} = W_\mu dx^\mu + S_\alpha dz^\alpha + S_{\dot\alpha} d\bar{z}^{\dot\alpha}$.

gauge sym: $\epsilon = \epsilon(x \mid Y, Z)$.

$$\delta\mathcal{A} = \underset{\hat{d} = d_x + d_Z}{\hat{d}\epsilon} + \mathcal{A} * \epsilon - \epsilon * \mathcal{A}$$

$$F = \hat{d}\mathcal{A} + \mathcal{A} * \mathcal{A}, \qquad \delta F = [F, \epsilon]_*$$

$$\delta B = -\epsilon * B + B * \pi(\epsilon).$$

"twisted adjoint rep" of $*$-alg.

π is defined by

$$\pi(f(y, z, dz, \bar{y}, \bar{z}, d\bar{z})) = f(-y, -z, -dz, \bar{y}, \bar{z}, d\bar{z})$$

can define $\bar{\pi}$ that flips the signs of anti-hol twistor variables,

But we will restrict to function invariant under $\pi\bar{\pi}$,

so $\pi \sim \bar{\pi}$.

[i.e. $W_\mu(x|Y,Z) = W_\mu(x|-Y,-Z)$.

$S_{\alpha\dot{\alpha}}(x|Y,Z) = -S_{\alpha\dot{\alpha}}(x|-Y,-Z)$.

$B(x|Y,Z) = B(x|-Y,-Z)$.]

Vasiliev's equations of motion

$$\begin{cases} \hat{d}\mathcal{A} + \mathcal{A}*\mathcal{A} = f(B*K)\,dz^2 + \bar{f}(B*\bar{K})\,d\bar{z}^2. \\ \hat{d}B + \mathcal{A}*B - B*\pi(\mathcal{A}) = 0. \end{cases}$$

not an independent eqn. in fact. - follows from Bianchi id.

$f(X)$ - a holomorphic $*$-function.

[Convention: $z^\alpha = \epsilon^{\alpha\beta} z_\beta$.

$dz^2 = \frac{1}{2} dz^\alpha \wedge dz_\alpha$]

i.e. $f(B*K) = a_0 + a_1 B*K + a_2 B*K*B*K + \dots$

$K = e^{z^\alpha y_\alpha}$, $\bar{K} = e^{\bar{z}^{\dot{\alpha}} \bar{y}_{\dot{\alpha}}}$ "Kleinian operator"

Can show : up to field redefinition.

$$B \to g(B*K)*K.$$

↖ real, odd $*$-function

$$\hat{S} := (\tfrac{1}{2} z_\alpha + S_\alpha)\, dz^\alpha + \text{c.c.}$$

$$\hat{S} \to \hat{S} * h(B*K)$$

↖ real, even $*$-function

can put $f(X)$ in the form

$$f(X) = X e^{i\theta(X)}, \qquad \theta(X) \text{ real, even.}$$

↖ product will be replaced by $*$-product.

<u>Parity</u> : $y_\alpha \leftrightarrow \bar{y}_{\dot\alpha}$, $z_\alpha \leftrightarrow \bar{z}_{\dot\alpha}$.

$$f(B*K)\, dz^2 \longleftrightarrow \bar{f}(B*\bar{K})\, d\bar{z}^2.$$

may assign B to be parity <u>even</u> or <u>odd</u>

$f(X) = f^*(X)$. $\qquad\qquad f(X) = f^*(-X)$

$\Downarrow \qquad\qquad \Downarrow$

$f(X) = X$. $\qquad\qquad f(X) = iX$.

A-type theory. $\qquad\qquad$ B-type theory.

[These are "non-minimal" th., turn out to have $s = 0, 1, 2, 3, \cdots$ in AdS_4]

Eqns admit the following truncation:

consider the operation $\iota_\pm$:

$$\iota_\pm : (y, \bar{y}, z, \bar{z}) \longmapsto (iy, \pm i\bar{y}, -iz, \mp i\bar{z}).$$

together with reversal of the order of $*$-product.

This preserves $*$-algebra,

i.e. $$\iota_\pm (F * G) = \iota_\pm (G) * \iota_\pm (F),$$

$[\iota_\pm^2 = \mathrm{id}$ by our truncation condition $\pi\bar{\pi}$-inv$]$.

Note: $$\iota_+ (B * K) = K * \iota_+ (B) = \iota_-(B) * K.$$

$$\left[K * f(y,z) * K = f(-y,-z) \right]$$
$$K * K = 1.$$

Consistent truncation;

$$\begin{cases} \iota_+(W) = -W, \\ \iota_+(S) = -S. \\ \iota_-(B) = B \end{cases}$$

$\Rightarrow$ minimal Vasiliev th.

$(s = 0, 2, 4, \ldots$ in $AdS_4)$

Where is the AdS ?

So far, eqns are bkgrd independent.

<u>AdS$_4$ vacuum sol'n</u> ;

$$\begin{cases} \mathcal{A}_0 = W_0(x|Y) = e_0(x|Y) + \omega_0(x|Y) \\ \quad = (e_0)_{\alpha\dot\beta}(x)\, y^\alpha \bar{y}^{\dot\beta} + (\omega_0)_{\alpha\beta}\, y^\alpha y^\beta + (\omega_0)_{\dot\alpha\dot\beta}\, \bar{y}^{\dot\alpha}\bar{y}^{\dot\beta} . \\ B = 0 . \end{cases}$$

$d_x W_0 + W_0 * W_0 = 0$ satisfied if e_0, ω_0 are vierbein and spin connection of AdS$_4$.

e.g. in Poincaré coord.

not to be confused w/ twistor variables

$$ds^2 = \frac{d\vec{x}^2 + dz^2}{z^2} , \qquad x_\mu = (\vec{x}, z)$$

$$e_0 = -\frac{1}{4}\frac{dx_\mu}{z}\, y\sigma_\mu\bar{y} = \frac{1}{4}\frac{dx_\mu}{z}\,\sigma^\mu_{\alpha\dot\beta}\, y^\alpha\bar{y}^{\dot\beta}$$

$$\omega_0 = \frac{1}{8}\frac{dx^i}{z}\left[(\sigma^{iz})_{\alpha\beta}\, y^\alpha y^\beta + (\sigma^{iz})_{\dot\alpha\dot\beta}\,\bar{y}^{\dot\alpha}\bar{y}^{\dot\beta}\right.$$

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Physical spectrum ?

i.e. propagating d.o.f.

Linearize around AdS_4 vacuum.

- will discover a tower of HS gauge fields propagating in AdS_4.

$$W = W_0 + \hat{W}, \quad S, \quad B.$$

$D_0 \equiv d_x + [W_0, \cdot]_*$,

$\widetilde{D}_0 \equiv d_x + W_0 * \cdot - \cdot * \pi(W_0)$,

(A-type th.)

$$D_0 \hat{W} = -\hat{W} * \hat{W}$$

$$\begin{cases} d_z \hat{W} = -D_0 S - \{\hat{W}, S\}_* \\ d_z S = B * (K\, dz^2 + \bar{K}\, d\bar{z}^2) - S * S \\ \widetilde{D}_0 B = -\hat{W} * B + B * \pi(\hat{W}) \\ d_z B = -S * B + B * \pi(S) \end{cases}$$

$\Downarrow$ linearize.

$\hat{W} = \Omega + W', \quad W'|_{Z=0} = 0$

$$D_0 \hat{W} = 0 \quad \xrightarrow{|_{Z=0}} D_0 \Omega = -(D_0 W')|_{Z=0}$$

fixed by B.

$$\begin{cases} d_z \hat{W} = -D_0 S \\ d_z S = B * (K dz^2 + \bar{K} d\bar{z}^2) \\ \widetilde{D}_0 B = 0 \\ d_z B = 0 \Rightarrow B = B(x|Y) \end{cases}$$

solve z-dept part of $\hat{W}$, i.e. W'

$S|_Z = 0$. gauge cond.

$\Downarrow$

$$S = \int^z B * K dz^2 + c.c.$$

All HS fields are contained in

$$\Omega = \hat{W}|_{Z=0}.$$

<u>and</u> $B = B(x|Y)$. ← $s=0$ is only here.

· expand $B(x|Y) = \sum B^{(n,m)}_{\alpha_1 \cdots \alpha_n \dot{\beta}_1 \cdots \dot{\beta}_m}(x)\, y^{\alpha_1} \cdots \bar{y}^{\dot{\beta}_1} \cdots$

and similarly $\Omega^{(n,m)}$ (which are 1-forms)

Now, we can answer "where are HS field?"

· Spin-s fields $\subset \Omega^{(s-1+k,\, s-1-k)}$, $B^{(2s+n,\, n)}$
$k = 1-s, 2-s, \cdots, s-1$, $B^{(n,\, 2s+n)}$
$n \geq 0$.

$$\Omega^{(s-1,s-1)}_{\underbrace{\alpha\dot{\beta}}_{\mu} | \alpha_1 \cdots \alpha_{s-1} \dot{\beta}_1 \cdots \dot{\beta}_{s-1}} \sim \Phi_{\alpha \alpha_1 \cdots \alpha_{s-1} \dot{\beta} \dot{\beta}_1 \cdots \dot{\beta}_{s-1}}$$

↑ double-traceless

gauge away trace part.

$$\Phi_{(\mu_1 \cdots \mu_s)}\, \sigma^{\mu}_{\alpha\dot{\beta}} \cdots \sigma^{\mu_s}_{\alpha_{s-1} \dot{\beta}_{s-1}}$$

rk-s symmetric traceless tensor field.

$B^{(2s,0)}_{\alpha_1 \cdots \alpha_{2s}}$, $B^{(0,2s)}_{\dot{\beta}_1 \cdots \dot{\beta}_{2s}}$ – spin-s generalization of Weyl tensor.

A chain of relations imposed by e.o.m.

$$\begin{tikzcd}
 & \Omega^{(s-2,s)} \arrow[dl, dotted] & \Omega^{(s-1,s-1)} \arrow[l] \arrow[r] & \Omega^{(s,s-2)} \arrow[dr, dotted] & \\
\Omega^{(0,2s-2)} \arrow[d] & & & & \Omega^{(2s-2,0)} \arrow[d] \\
B^{(0,2s)} \arrow[d] & & & & B^{(2s,0)} \arrow[d] \\
B^{(1,2s+1)} \arrow[d] & & & & B^{(2s+1,1)} \arrow[d] \\
\vdots & & & & \vdots
\end{tikzcd}$$

neighboring 3
are related by

$$D_0 \Omega = -(D_0 W')\big|_{Z=0}.$$

Simplest example : the $s=0$ component.

- scalar field, contained only in $B^{(n,n)}$.

Look at linearized eqn. $\tilde{D}_0 B(x|Y) = 0$.

!

i.e.

$$d_x B + [\omega_0, B]_* + \{e_0, B\}_* = 0$$

$$\Downarrow$$

$$\nabla_\mu B^{(n,m)} + 2(e_0)_\mu{}^{\alpha\dot\beta} \left(y_\alpha \bar{y}_{\dot\beta} B^{(n-1,m-1)} + \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \bar{y}^{\dot\beta}} B^{(n+1,m+1)} \right) = 0.$$

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Lowest components

$$\partial_\mu B^{(0,0)} + 2 e_\mu{}^{\alpha\dot\beta} B^{(1,1)}_{\alpha\dot\beta} = 0 .$$

$$\nabla_\mu B^{(1,1)}_{\alpha\dot\beta} + 2(e_\mu)_{\alpha\dot\beta} B^{(0,0)} + 2 e_\mu{}^{\gamma\dot\delta} B^{(2,2)}_{\alpha\gamma\dot\beta\dot\delta} = 0 ,$$

etc.

$$4 \nabla_{\alpha\dot\beta} B^{(0,0)} + 2 B^{(1,1)}_{\alpha\dot\beta} = 0 ,$$

$$\underbrace{4 \nabla_{\gamma\dot\delta} B^{(1,1)}_{\alpha\dot\beta} + 2 \epsilon_{\gamma\alpha} \epsilon_{\dot\delta\dot\beta} B^{(0,0)} + 2 B^{(2,2)}_{\alpha\gamma\dot\beta\dot\delta} = 0}_{\text{contract with } \epsilon^{\alpha\gamma}\epsilon^{\dot\beta\dot\delta}}$$

[our convention of $e_\mu{}^{\alpha\dot\beta}$ is such that

$$4 e_\mu{}^{\alpha\dot\beta} \nabla_{\alpha\dot\beta} = \nabla_\mu ,$$

$$\nabla_{\alpha\dot\beta} = -\frac{1}{8} e^\mu{}_{\alpha\dot\beta} \nabla_\mu .$$

$$-2 \nabla^{\alpha\dot\beta} \nabla_{\alpha\dot\beta} = \nabla^\mu \nabla_\mu .$$]

$$\Rightarrow \quad \nabla^{\alpha\dot\beta} B^{(1,1)}_{\alpha\dot\beta} + 2 B^{(0,0)} = 0 .$$

$$\Downarrow$$

$$(\nabla^\mu \nabla_\mu - m^2) B^{(0,0)} = 0, \qquad m^2 = -2 \quad \text{(AdS radius set to 1)}$$

☆ Relating $\Omega \ (= \hat{W}|_{Z=0})$ to $B(x|Y)$.

at linearized level:

$$(D_0 \hat{W})\Big|_{Z=0} = 0 , \qquad \hat{W} = \Omega + W' , \qquad W'|_Z = 0 .$$

$$\Omega = \Omega(x|Y).$$

$$\Downarrow$$

$$D_0 \Omega = -(D_0 W')\Big|_{Z=0} = -\left(\underbrace{dW'}_{\to 0 \text{ after setting } Z=0} + \{W_0(x|Y), W'\}_*\right)\Big|_{Z=0}$$

$$= -\underbrace{\{W_0, W'\}_*}_{y,z \text{ contract}}\Big|_{Z=0} \sim \left\{ \frac{\partial W}{\partial y^\alpha}, \frac{\partial W'}{\partial z_\alpha}\Big|_{Z=0} \right\}_*$$

only need linear term in z_α in W

What is W'? ← determined by S, and hence by B, via

$$d_z W' = -D_0 S, \quad \ldots,$$

can derive rel'n b/t Ω and B:

$$D_0 \Omega = \frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial y^\beta} B(x|y, \bar{y}=0)\, e^{\alpha\dot{\gamma}} \wedge e^{\beta}{}_{\dot{\gamma}} + \text{c.c.}$$

for spin-s fields, only $B^{(2s,0)}$ enters here, and $B^{(0,2s)}$

Using the gauge sym

$$\delta\Omega = d_x \epsilon + [W_0, \epsilon]_*,$$

restricted to $\epsilon(x|Y)$,

can partially gauge fix, such that the entire Ω is fixed in terms of $\Omega^{(s-1,s-1)}$ via a rel'n of the form

$$\Omega = \left(1 + \sum_{n=1}^{s-1} \hat{T}_+^n + \sum_{n=1}^{s-1} \hat{T}_-^n\right) \Omega^{(s-1,s-1)}.$$

$\hat{T}_+$ raises degree in y, lowers deg in $\bar{y}$,

$\hat{T}_-$ does the opposite;

both are 1^{st} order diff. operators on x^μ.

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Simplest example. $s=1$, vector gauge field
(in non-minimal Vasiliev th.)

$$\Omega^{(0,0)} = \Omega^{(0,0)}_{\mu}(x)\, dx^{\mu}.$$

$\uparrow A_\mu$.

$$D_0\Omega^{(0,0)} = dA = F^+ + F^-$$

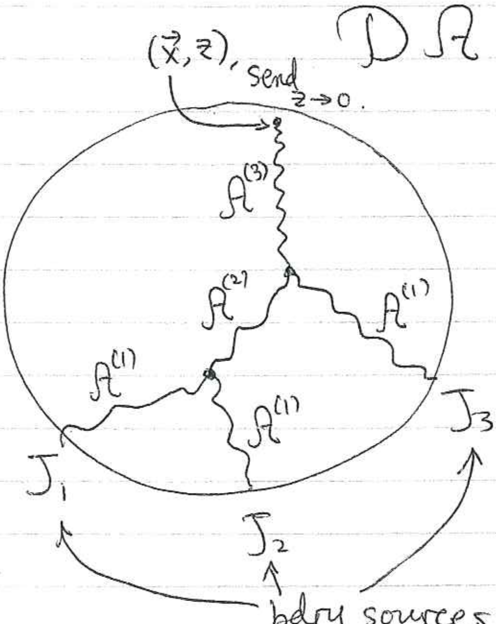
$$F^+ \leftrightarrow B^{(2,0)}_{\alpha\beta} \qquad F^- \leftrightarrow B^{(0,2)}_{\dot\alpha\dot\beta}.$$

Now, let us do perturbation theory.

At the level of master fields.

e.o.m. skhematically of the form

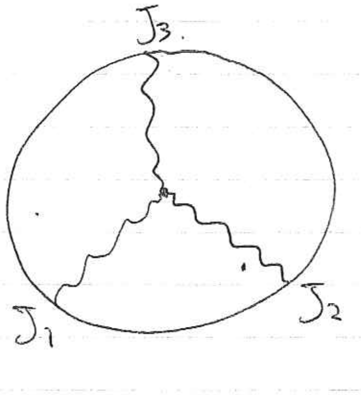
$$D\Omega = \Omega * \Omega$$



To compute bdry n-point correlators.
[GKP, Witten].

Can take $(n-1)$ of them as bdry sources, solve the $(n-1)$-th order bulk field by integrating the e.o.m. iteratively, and then send this bulk field to bdry to extract its bdry limiting value $\propto$ n-pt function.

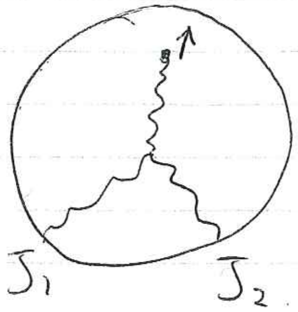
Three-Point Function



Even cubic vertex difficult to extract directly from Vasiliev's equations

[in principle straightforward, but a horrific mess]

But, can compute directly using e.o.m.



given bdry source $J^{(s)}_{i_1 \cdots i_s}(\vec{x}_0)$.

$\updownarrow$

polarization tensor.

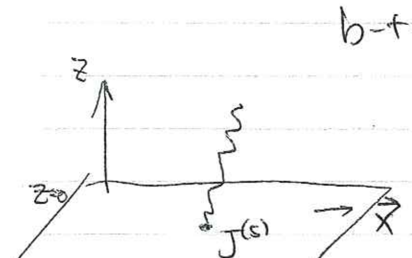
$$\varepsilon_{i_1 \cdots i_s} = \varepsilon_{i_1} \cdots \varepsilon_{i_s}$$

$\uparrow$ WLOG assumed to be a <u>null</u> vector.

Solve boundary-to-bulk propagator for B, Ω at <u>linear</u> order

e.g. a spin-s source at $\vec{x}=0$, polarization ε_i $\rightarrow \varepsilon_{\alpha\beta} = \lambda_\alpha \lambda_\beta$.

b-t-b propagator for B.



$$B(x|Y) = \frac{z^{s+1}}{(\vec{x}^2+z^2)^{s+1}} \, e^{-y^\alpha \Sigma_\alpha{}^{\dot\beta} \bar{y}_{\dot\beta}} \, (y \mathbf{X} \bar{\lambda})^{2s} + \text{c.c.}$$

$$\mathbf{X} = x^\mu \sigma_\mu, \quad \Sigma = \sigma^z - \frac{2z}{\vec{x}^2+z^2}\mathbf{X}, \quad \bar{\lambda} = \sigma^z \lambda.$$

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Having solved 1st order fields from bdry sources, $B^{(1)}, S^{(1)}, W^{(1)}$,

solve for $B^{(2)}$ from

$$\tilde{D}_0 B^{(2)} = -\hat{W}^{(1)} * B^{(1)} + B^{(1)} * \pi(\hat{W}^{(1)}) .$$

$\uparrow$

$B^{(2)}(x|Y,Z)$,

let $C(x|Y) = B\big|_{Z=0}$. $\quad B = C + \check{B}$.

$\check{B}^{(2)}$ solved using

$$d_Z \check{B} = -S * B + B * \pi(S) .$$

$$\tilde{D}_0 B^{(2)}\Big|_{Z=0} = \tilde{D}_0 C^{(2)} + \underbrace{(\tilde{D}_0 \check{B})\Big|_{Z=0}}_{\substack{\shortparallel \\ -J^Z}} = \left[-\hat{W}^{(1)} * B^{(1)} + B^{(1)} * \pi(\hat{W}^{(1)})\right]\Big|_{Z=0} \equiv J^Y .$$

$$\Rightarrow \tilde{D}_0 C = J^Y + J^Z \equiv J_\mu dx^\mu .$$

$\leftarrow$ calculate from b-t-b propagators and $*$-product.

algebraic manipulation

$$(\Box + \cdots)\, C^{(2s,0)}(x|y) = J(y) .$$

Solve $C^{(2s,0)}(x|y)$.

$\to$ self-dual part of HS field strength. $\longrightarrow$ fake $z \to 0$ limit

Note: $z \to 0$, limit,

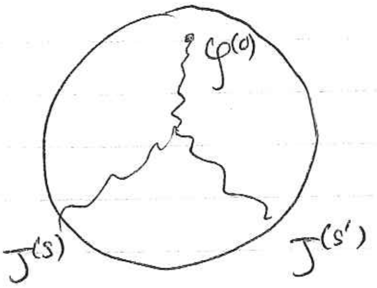
$$C^{(2s,0)}_{\alpha_1\alpha_2\cdots\alpha_{2s}}(\vec{x}, z) \quad \text{dominated by}$$

$$\partial_z^s \varphi_{i_1\cdots i_s}(\vec{x}, z).$$

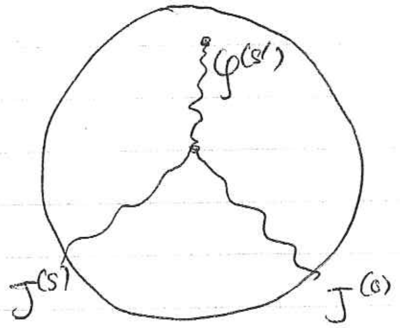
To extract $\langle \varphi_{i_1\cdots i_s} \rangle_{z\to 0}$, enough to look at $C^{(2s,0)}$

In 0912.3462 (w/ S. Giombi),

did this computation explicitly for



and



$s=s'$ case singular... much like $(\text{scalar})^3$, $0\times\infty$ problem.

need $s>s'$ to avoid gauge ambiguity.

Results agree w/ crossing symmetry.

i.e. $\langle J_1 J_2 J_3 \rangle$,

can regard any two as sources,

third J from expectation value of "outgoing" field.

Normalization precisely agree with 3-pt fn in free $O(N)$ vector model. ✓

A simplified method

"gauge function method",

or "$W=0$ gauge".

· Part of Vasiliev's eqns $\Rightarrow$

$$d_x W + W * W = 0.$$

i.e. W is a flat connection in x^μ

Can write (at least locally)

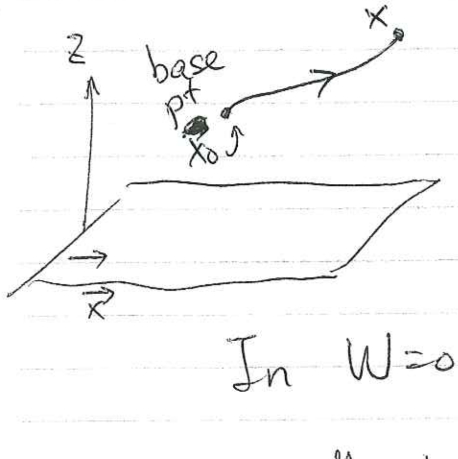
$$W = L^{-1} * dL.$$

L^{-1}: $*$-inverse of L.

in particular,

$$W_0 = L_0^{-1} * dL_0,$$

$$L_0 = P \exp_* \left(- \int_x^{x_0} W_0^\mu (x'|Y)\, dx'_\mu \right).$$



explicit form of L_0 known.

In $W=0$ gauge, $d_x S = d_x B = 0$.

x^μ-dependence drops out entirely in S, B.

call these

$$S'(Y,Z), \qquad B'(Y,Z).$$

In this gauge, Vasiliev's system reduces to simply: (A-type th.)

$$\begin{cases} d_Z S' + S' * S' = B' * (K dz^2 + \bar{K} d\bar{z}^2) \\ d_Z B' + S' * B' - B' * \pi(S') = 0. \end{cases}$$

To go back to "physical" gauge:

$$W = L^{-1} * d_x L$$

$$S = L^{-1} * d_Z L + L^{-1} * S' * L.$$

$$B = L^{-1} * B' * \pi(L).$$

Note: $B' = B\big|_{x = x_0,}$ base pt. $L(x_0) = 1$.

So, already know b-t-b propagator in $W=0$ gauge,

Solving Vasiliev's eqns in pert. th is then a simple exercise of integrating in Z $= (z_\alpha, \bar{z}_{\dot\alpha})$.

Finally transform back to physical gauge, take $z \to 0$ limit

I will not discuss the details of this calculation here, but to point out a few subtleties.

- read off 3-pt fn.

$$\langle J^{(s)}(\vec{x}, \lambda)\, J^{(s_1)}(x_1, \lambda_1)\, J^{(s_2)}(x_2, \lambda_2) \rangle$$

polarization spinor

$$= \lim_{z \to 0} z^{-1} B^{(2)}(\vec{x}, z \mid y_\alpha = z^{-\frac{1}{2}} \lambda_\alpha,\ \bar{y}_{\dot\alpha} = z_\alpha = \bar{z}_{\dot\alpha} = 0).$$

Poincaré radial coord

- Gauge ambiguity

going from $B'^{(2)}(Y, Z)$ to physical gauge via L.

$$B^{(2)}(x \mid Y, Z) = L_0^{-1}(x \mid Y) * B'^{(2)}(Y, Z) * \pi(L_0(x \mid Y))$$
$$- \epsilon^{(1)}(x \mid Y, Z) * B^{(1)}(x \mid Y) + B^{(1)}(x \mid Y) * \pi(\epsilon^{(1)}(x \mid Y, Z$$

1st order gauge ambiguity.

$$W^{(1)}(x \mid Y, Z) = D_0 \epsilon^{(1)}(x \mid Y, Z).$$

Also need $S|_{Z=0} = 0$.

Demand certain $z \to 0$ fall off condition on allowed $\epsilon^{(1)}(x \mid Y, Z)$.

Such allow gauge transformations turn out to not affect result of bdry 3-pt function, provided

$$s < s_1 + s_2$$

spin of outgoing field

spins of sources.

- Encounter singular contour integrals in computing $*$ - product

  (the class of functions involved in b-t-b propagators of master fields do not always behave nicely under $*$ - product)

  regularize by deforming contour.

  - obtain results that agree with physical gauge computation in previously computed cases.
  - don't know general contour prescription or why it works.
  - contour prescription can be stated as contour on "twistor space". Fourier transform half of the twistor variables (y_1 or $\bar{y}_2$).

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Result: A generating function for 3-pt Correlators.

$$\sum_{s_1, s_2, s_3} \left\langle J^{(s_1)}(x_1, \lambda_1)\, J^{(s_2)}(x_2, \lambda_2)\, J^{(s_3)}(x_3, \lambda_3) \right\rangle$$

$$= \frac{1}{|x_{12}|\cdot|x_{23}|\,|x_{31}|} \cosh\left(\frac{Q_1 + Q_2 + Q_3}{2}\right) \prod_{i=1}^{3} \cosh P_i \,,$$

$$P_1 = \lambda_2 \check{x}_{23} \lambda_3 \,, \qquad \check{x}_{ij} = \frac{x_{ij}}{x_{ij}^2} \,, \qquad \text{cyclicly},$$

$$Q_1 = -\lambda_1 (\check{x}_{12} + \check{x}_{31}) \lambda_1 .$$

This expression was first computed from A-type Vasiliev th., in 1004.3736, then shown to agree exactly with free $O(N)$ theory!

In B-type Vasiliev thy, in this computation of 3-pt fn, only difference is in b-to-b propagator for B.

Convient to Fourier transform external polarization spinor λ.

$$B'^{(1)}(Y;\lambda) \rightsquigarrow B^{(1)}_{tw}(Y;\mu) = \frac{1}{4}\int d^2\lambda\, e^{2\lambda\mu} B'^{(1)}(Y;\lambda)$$

A-type th:

$$B^{(1)}_{tw}(Y;\mu) = \delta(y-x)\, e^{\bar{x}\bar{y}} + \delta(\bar{y}-\bar{x})\, e^{xy}.$$

B-type th.

$$B^{(1)}_{tw}(Y;\mu) = i\,\delta(y-x)\, e^{\bar{x}\bar{y}} - i\,\delta(\bar{y}+\bar{x})\, e^{xy}.$$

["b-t-b propagators are δ-fns on twistor space!"].

3-pt fn.

$$\frac{1}{|x_{12}|\,|x_{23}|\,|x_{31}|} \cdot \sinh\left(\frac{Q_1+Q_2+Q_3}{2}\right) \cdot \prod_{i=1}^{3} \sinh P_i,$$

match precisely that of free fermions

Why are these nontrivial tests of the bulk theory?

General constraints from <u>conformal</u> invariance + conservation of currents.

$$\Rightarrow \langle J^{(s_1)} J^{(s_2)} J^{(s_3)} \rangle$$

$$= a_1 \underset{\substack{\uparrow \\ \text{free scalar}}}{\langle J J J \rangle_B} + a_2 \langle J J J \rangle_F \leftarrow \text{free fermion}$$

$$+ \, b \langle J J J \rangle_{odd}.$$

↗ parity odd, exists only when 

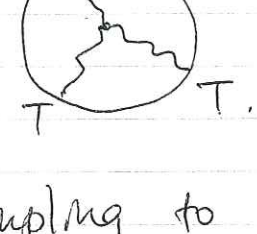
is obeyed.

e.g. bulk graviton $\longleftrightarrow$ stress-energy tensor T_{ij}

$$\langle T T T \rangle = a_1 \langle T T T \rangle_B + a_2 \langle T T T \rangle_F$$

in parity-inv't th.

Einstein-Hilbert action

$$\int_{AdS_4} \sqrt{-g}\,(R - \Lambda) \rightsquigarrow$$ 

= a linear combination of $\langle TTT \rangle_B$ and $\langle TTT \rangle_F$

Need higher deriv. coupling to match w/ just $\langle TTT \rangle_B$ (or $\langle TTT \rangle_F$).

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We have seen that Vasiliev th. has precisely the correct higher derivative couplings (required by HS sym) to produce the free 3-pt functions.

☆ Now, what about the critical $O(N)$ model?

- A-type th. w/ $\Delta=2$ boundary condition.
- HS sym broken by $1/N$ effects.
- only modification in the bulk is the propagator of the spin-0 field $\varphi^{(0)}$.

Scalar propagator in AdS_4 $\left(m^2=-\frac{2}{R^2}\right)$. $R=1$.

$$G_\Delta(x,x') = \frac{1}{4\pi^2}\frac{\xi^\Delta}{1-\xi^2}, \qquad \xi = \frac{1}{\cosh d(x,x')}$$

$\Delta=1,2$. geodesic distance.

same singularity at $x=x'$, different IR behavior.

$x=(\vec{x},z)$, Fourier transform $\vec{x}\to\vec{p}$.

$\rightsquigarrow G_\Delta(p\,;z,z')$

Then,

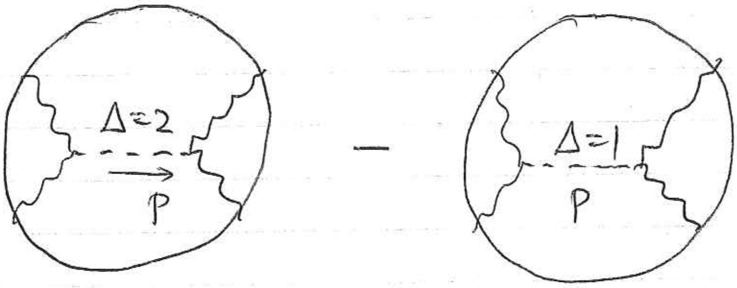
$$G_{\Delta=2}(p;z,z') - G_{\Delta=1}(p;z,z')$$

$$= -|p| \cdot K_{\Delta=1}(p;z)\, K_{\Delta=1}(p;z).$$

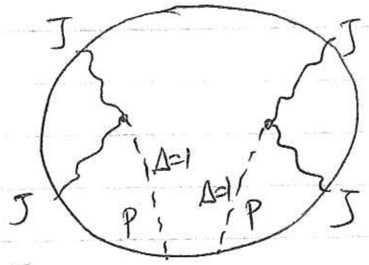
$$K_{\Delta=2}(p,z) = -|p|\, K_{\Delta=1}(p,z) = z\, e^{-|p|z}.$$

↖ boundary-to-bulk propagator.

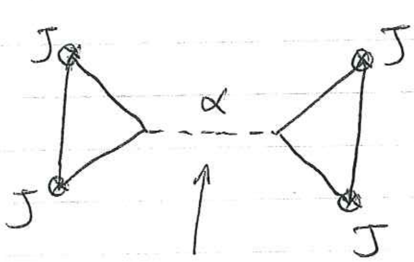
Diagrammatically,



= −|p| ×



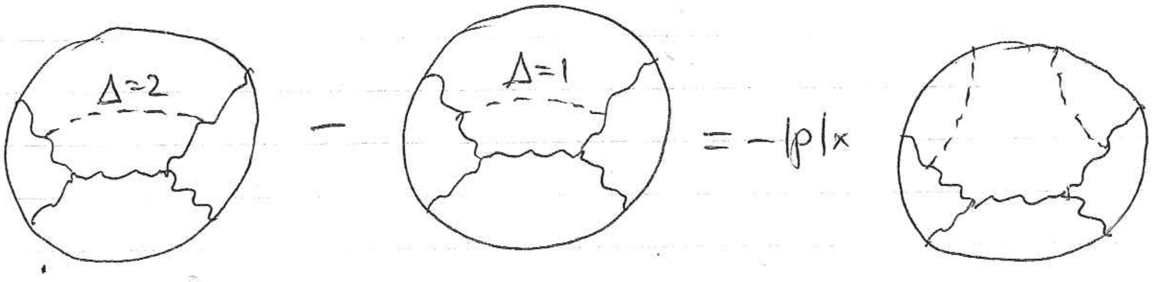
in CFT 1/N expansion, this is



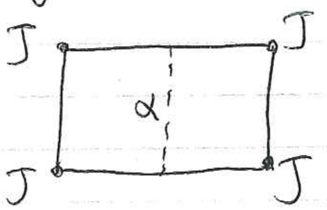
−|p| is α-propagator.

Also works at loop-level.

e.g.



such contributions correspond to diagrams like



in critical O(N) model.

Can show, to <u>all order</u> in perturbation theory (in $1/N$)

if Vasiliev A-type th. w/ $\Delta=1$ b.c.

$=$ free O(N) th., ← [implies all $1/N$ corr. w/ $\Delta=1$ b.c cancel in the bulk th.

then, A-type th w/ $\Delta=2$. b.c.

$=$ critical O(N) vector model.

Breaking HS sym by boundary condition.

- Anomalous current conservation (ACC) relations in critical O(N) model.

$$\mathcal{L} = N\left[\frac{1}{2}(\partial_\mu \vec{\phi})^2 + \frac{1}{2}\alpha\left(\vec{\phi}^2 - \underbrace{\frac{1}{g}}_{\to 0}\right)\right]$$

$\to 0$ @ criticality

$\vec{\phi}^2$ is no longer an operator in this description.

$J^{(0)} = \alpha$. $\quad \Delta = 2 + \mathcal{O}(\frac{1}{N})$.

eqn of motion for ϕ_i:

$$\Box \vec{\phi} = \alpha \vec{\phi}$$

$$J^{(s)}(x, \varepsilon) \equiv J^{(s)}_{\mu_1 \cdots \mu_s}(x)\, \varepsilon^{\mu_1} \cdots \varepsilon^{\mu_s}$$

$\leftarrow$ null vector

$$= \phi_i\, f(\varepsilon\cdot\vec{\partial}, \varepsilon\cdot\overleftarrow{\partial})\, \phi_i \Big|_{\varepsilon^s}$$

$$f(u, v) = e^{u-v} \cos(2\sqrt{uv}).$$

would be conserved if $\Box\vec{\phi} = 0$,

Now, we have instead

$$\partial^\mu J^{(s)}_{\mu\mu_1 \cdots \mu_{s-1}} = \sum_{\substack{s' < s \\ n+m+s' = s-1}} \partial^n \alpha\, \partial^m J^{(s')}$$

Normalization is s.t.

$$\langle \alpha \alpha \rangle \sim \frac{1}{N}, \quad \langle J J \rangle \sim N$$ so far.

For operators w/ normalized 2-pt fn.

$$\partial^\mu \left(\frac{1}{\sqrt{N}} J^{(s)}_{\mu \ldots} \right) \sim \frac{1}{\sqrt{N}} \sum_{s'<s} \partial^n (\sqrt{N} \alpha) \cdot \partial^m \left(\frac{1}{\sqrt{N}} J^{(s')}_{\ldots} \right)$$

$J^{(s)}$ not exactly conserved, acquires anomalous dim of order $\frac{1}{N}$.

primary of conformal algebra

$$\| P^\mu | J_{\mu \ldots} \rangle \|^2 = \langle J_{\nu \ldots} | K^\nu P^\mu | J_{\mu \ldots} \rangle$$

$$= \langle J_{\nu \ldots} | \underbrace{[K^\nu, P^\mu]}_{= \delta^{\mu\nu} D - M^{\mu\nu}} | J_{\mu \ldots} \rangle. \quad \propto (\Delta - s - 1).$$

$$\to \frac{1}{N} \sum \langle \alpha | \alpha \rangle \langle J^{(s')} | J^{(s')} \rangle \quad \Rightarrow \quad \Delta = s + 1 + \mathcal{O}\left(\frac{1}{N}\right).$$

Now, consider 3-pt fn.

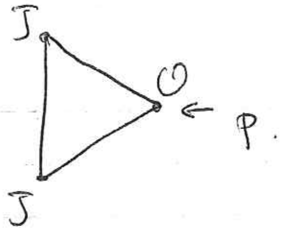
$$\langle J^{(s)} J^{(s')} \alpha \rangle, \quad s' < s.$$

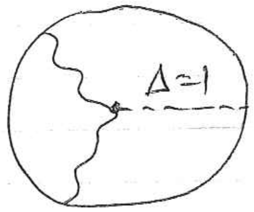
$$\langle \partial \cdot J^{(s)} J^{(s')} \alpha \rangle \sim \langle \alpha \alpha \rangle \langle J^{(s')} J^{(s')} \rangle.$$

$\neq 0$ already at <u>leading</u> order in $\frac{1}{N}$!

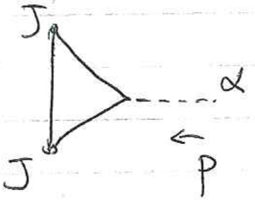
i.e. HS sym violated at <u>bulk</u> tree level, how come?

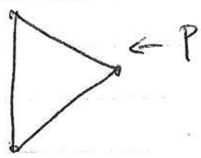
In momentum space.

free $O(N)$ th: $\langle J^{(s)} J^{(s')} \mathcal{O}(p) \rangle =$ 



critical $O(N)$, leading order in $1/N$:

$\langle J^{(s)} J^{(s')} \alpha(p) \rangle =$ 

$= -|p| \times$ 

$s > s'$,

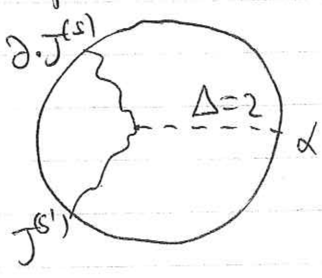
$\langle \partial \cdot J^{(s)} J^{(s')} \mathcal{O} \rangle$ is a contact term

i.e. analytic in momentum p (at $p=0$)

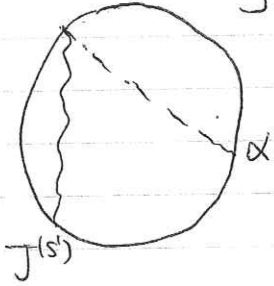
$\langle \partial \cdot J^{(s)} J^{(s')} \alpha \rangle$ is not analytic in p @ $p=0$.

Bulk picture:

bdry term contribution.



$\longrightarrow$



Generalizations of the vector model

- free $U(N)$, or $SU(N)$ vector model

  currents of spin $s=0, 1, 2, 3, \cdots$

  dual to non-minimal Vasiliev th. in AdS_4,

  A / B-type for scalar / fermion dual.

Now, we can choose different boundary conditions on bulk vector $^{s=1}$ gauge field A_μ.

[Witten '03].

ordinary bdry condition

$$A_\mu \longleftrightarrow U(1) \text{ current } J_i \text{ in CFT.}$$

$$F_{ij}\big|_{z=0} = 0, \quad F_{zi}\big|_{z=0} \text{ unconstrained.}$$

E-M dual bdry condition.

$$F_{zi}\big|_{z=0} = 0, \quad F_{ij}\big|_{z=0} \text{ unconstrained}$$

$\longleftrightarrow$ gauging $U(1)$ current J_i on bdry.

$S \to S + \int a_i J^i$. integrating out "matter fields" generate propagator for a_i.

This is the IR critical pt of

U(1) QED + N flavors (scalar or fermion)

$\leftarrow F^a_{\mu\nu} F^{a\mu\nu}$ irrelevant in the IR,

dominated by $\langle J_i J_j \rangle$.

There is more:

mixed bdry condition can be conformally invt:

$$\left(\frac{1}{2}\epsilon_{ijk} F_{jk} + \frac{N}{k} F_{zi}\right)\Big|_{z=0} = 0$$

$$\updownarrow$$

gauge U(1) current, w/ CS coupling @ level k

$$S \to S + \int a_i J^i + \frac{k}{4\pi}\int a \wedge da$$

If we use the E-M dual boundary condition

for $A_\mu = \varphi^{(1)}_\mu$, and $\Delta = 2$ b.c. on $\varphi^{(0)}$,

$\xleftrightarrow{\text{dual}}$ critical $\mathbb{CP}^{N-1}$ model.

$$\mathcal{L} = |D_\mu \phi_i|^2 + \alpha\left(|\phi_i|^2 - \frac{1}{g}\right) + \frac{1}{g^2_{YM}} F_{\mu\nu}F^{\mu\nu}$$

$\frac{1}{g} \to 0$, $\frac{1}{g^2_{YM}} \to 0$

IR fixed pt of NLσM on $\mathbb{CP}^{N-1} \cong S^{2N-1}/U(1)$

- matrix Vasiliev th.
  - replace $*$-algebra by its tensor product with the algebra of $M \times M$ matrices.

  "Chan-Paton factors"

  $\rightarrow$ $U(M)$ - HS gauge fields.

Dual: take $N \times M$ scalars ϕ_{ia}. (or fermions)

gauge $\underset{\text{cplx}}{U(N)}$ or $\underset{\text{real}}{O(N)}$.

residual $U(M)$ or $O(M)$ global sym

dual to bulk $\overset{\text{nonabelian}}{\text{vector}}$ gauge fields.

If we change bdry condition.

$\rightarrow$ e.g. generalization of critical $O(N)$,

or critical $U(M)$ QCD w/ N flavors ...

colored graviton in the bulk confine

due to bdry condition, only one conserved stress-energy tensor left.

- Chern-Simons Vector Model.

SU(N) CS at level k + $\square$ massless matter,

say Dirac fermion ψ.

$$S_E = \frac{ik}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$
$$+ \int \bar{\psi}\not{D}\psi.$$

- non-supersymmetric, exact CFT with Lagrangian description.
- [only non-irrelevant term $m\bar{\psi}\psi$, tune away.]

$k\to\infty$ limit, CS gauge fields become free, no propagating d.o.f.

$\rightarrow$ free SU(N) vector model

finite k - an interacting vector model.

't Hooft limit $N, k \gg 1$, $\lambda = \frac{N}{k}$ finite,

$\sim O(N)$ degrees of freedom

<u>Claim</u>: dual to a ~~parity~~ HSGT in AdS_4.

presumably, a parity violating Vasiliev th!

"Single trace" operators

$J^{(0)} = \bar{\Psi}\Psi$

$$\left.\begin{array}{l} J^{(1)}_{\mu} = \bar{\Psi}\gamma_\mu\Psi \\ J^{(2)}_{\mu\nu} = \bar{\Psi}\gamma_{(\mu}\overleftrightarrow{D}_{\nu)}\Psi \end{array}\right\} \text{exactly conserved}$$

$J^{(3)}_{\mu\nu\rho}$, etc. $\leftarrow$ conserved in free theory $(k\to\infty)$, or at $N=\infty$, but not at finite N.

HOWEVER, single trace operators acquire anomalous dim only at order $\frac{1}{N}$. for any finite λ.

Argument 1 in free theory, S.T.O. fall into reps of conformal algebra.

$$(\Delta, J) = \underset{(2,0)}{\bar{\Psi}\Psi} \leftarrow \text{long rep}$$

$$\left.\begin{array}{c} (2,1) \\ (3,2) \\ \vdots \\ (s+1,s) \end{array}\right\} \text{short rep.}$$

$\vdots$

To acquire anomalous dim. $\longrightarrow$ long reps.

$(s+1+\delta, s)$ long rep

$$\downarrow \delta \to 0$$

$(s+1, s) \oplus (s+2, s-1)$ ← no such rep among single trace operators!

$\hat{j}_{\mu\cdots} \qquad \partial^\mu \hat{j}_{\mu\cdots}$

$(s+1, s)$ must combine with some double-trace op. in order to be lifted at finite N.
- suppressed by $1/N$.

Argument 2.

Can show classically (analogous to critical $O(N)$ model)

$$\partial^\mu J^{(s)}_{\mu\mu_1\cdots\mu_{s-1}} = \frac{\lambda}{\sqrt{N}} \sum_{s_1+s_2<s} \partial^{\cdots} J^{(s_1)} \partial^{\cdots} J^{(s_2)} + \frac{\lambda^2}{N} \sum JJJ$$

(?)

normalized s.t. $\langle JJ \rangle \sim 1$.

can be renormalized to $f(\lambda)$ quantumly, but same order in N.

Anomalous dim

$$\delta \sim \langle \partial^\mu J_{\mu\cdots} | \partial^\nu J_{\nu\cdots} \rangle \sim \frac{1}{N} \langle JJ|JJ \rangle$$

This shows that $J^{(s)}$ for $s>0$ are not renormalized at $N=\infty$, finite λ.

What about $s=0$?

e.g. look at

$$\partial^\mu J^{(3)}_{\mu\nu\rho} \sim \sum \partial^{\cdot\cdot} J^{(0)} \partial^{\cdot\cdot} J^{(1)}$$

dimension of $J^{(0)}$ fixed by that of $J^{(3)}$ and $J^{(1)}$ at leading order in $1/N$.

Now, CS gauge fields don't carry propagating d.o.f.

$\rightarrow$ $O(N)$ d.o.f (rather than N^2).

exactly solvable ?

Yes, go to a gauge in which A^3 coupling goes away.

No.

DATE

1. temporal / axial gauge

$$A_3 = 0.$$

- troubling divergence at 2-loop and up.
- difficult to regularize.

2. lightcone gauge

$$A_- = 0, \qquad A_- \equiv \frac{A_1 + iA_2}{\sqrt{2}}$$

defined by analytic continuation from Lorentz signature.

in particular, A_+ still an independent field.

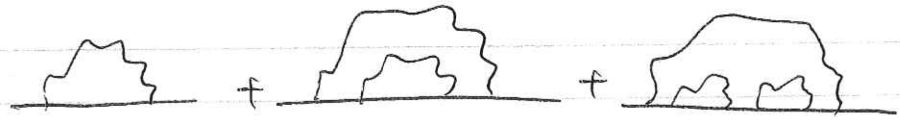
- easily regularized in dimensional reduction scheme!

- Need to check Lorentz inv. restored at quantum level in gauge inv't observables.
  - nontrivial, but works!

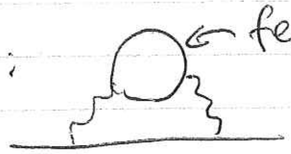
To solve <u>exact</u> fermion propagator at <u>large N</u> :

$$\langle \psi(p)\, \bar{\psi}(q) \rangle = \delta(p+q)\, \frac{1}{i\not{p} + \Sigma(p)}.$$

self-energy

$\Sigma(p)$ = 

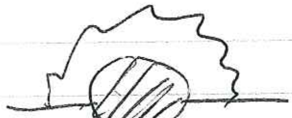
+ ....

Note:  ← fermion loop

is non-planar !

(unlike th. w/ adj matter)

At large N, have Schwinger-Dyson eqn. for fermion self-energy.

Σ = 

regularize: dimension $\to 3-\epsilon$, separate x^+, x^-.

take remaing 1 dim $\to (1-\epsilon)$-dim.

Result: $\Sigma(p) = \lambda |p_s| - i\lambda^2 p^- \gamma^+$

spatial momentum

$p_s^2 = 2p^+p^-$

2-loop exact!
(regularization scheme dependent)

Not manifestly Lorentz invt.
but also not a gauge invt quantity.

Look at poles of propagator:

$$\langle \psi \bar{\psi} \rangle = \frac{1}{i\not{p} - i\lambda^2 p^- \gamma^+ + \lambda |p_s|}$$

pole at

$$\underbrace{2(1-\cancel{\lambda^2}) p^- p^+ + p_3^2}_{= p^\mu p_\mu} + \cancel{\lambda^2} p_s^2 = 0.$$

pole structure indeed Lorentz invt!

Finite temperature : $S^1_\beta \times \mathbb{R}^2$,

can solve Schwinger-Dyson eqn.

$$\Rightarrow \Sigma_T(p) = f(\beta|p_s|)\cdot|p_s|\,\mathbb{1} + i\,g(\beta|p_s|)\,p^-\gamma^+.$$

$$f(y) = \frac{2\lambda}{y}\ln\left(2\cosh\frac{\sqrt{c+y^2}}{2}\right).$$

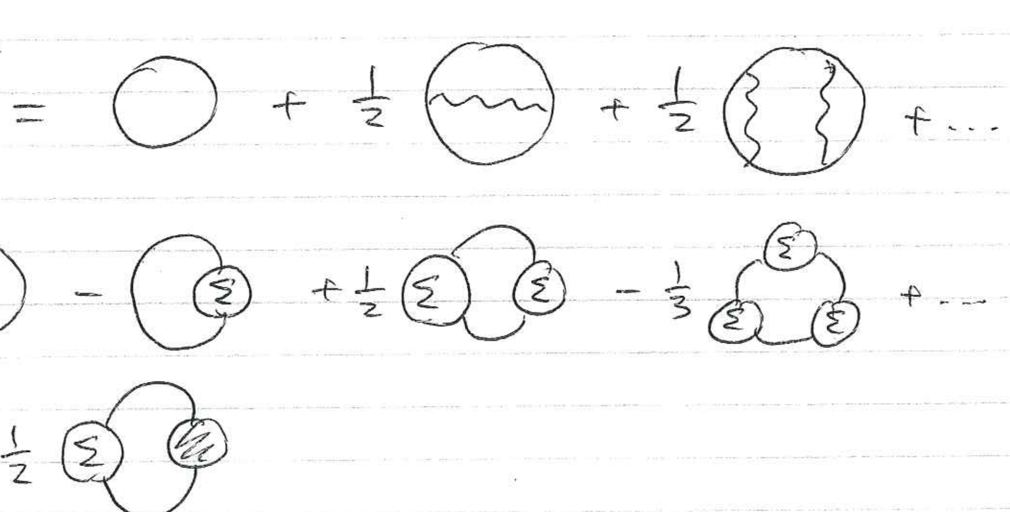
$$g(y) = \frac{c}{y^2} - f(y)^2.$$

c is determined by λ via rel'n

$$\frac{\sqrt{c}}{2\lambda} = \ln\left(2\cosh\frac{\sqrt{c}}{2}\right).$$

- Free energy at finite T

F^{planar}



$$= \mathrm{Tr}\ln(\not{\partial}+\Sigma) - \frac{1}{2}\mathrm{Tr}\left(\Sigma\cdot\frac{1}{\not{\partial}+\Sigma}\right).$$

can derive from path $\int$,

by integrating out A_μ in l.c. gauge.

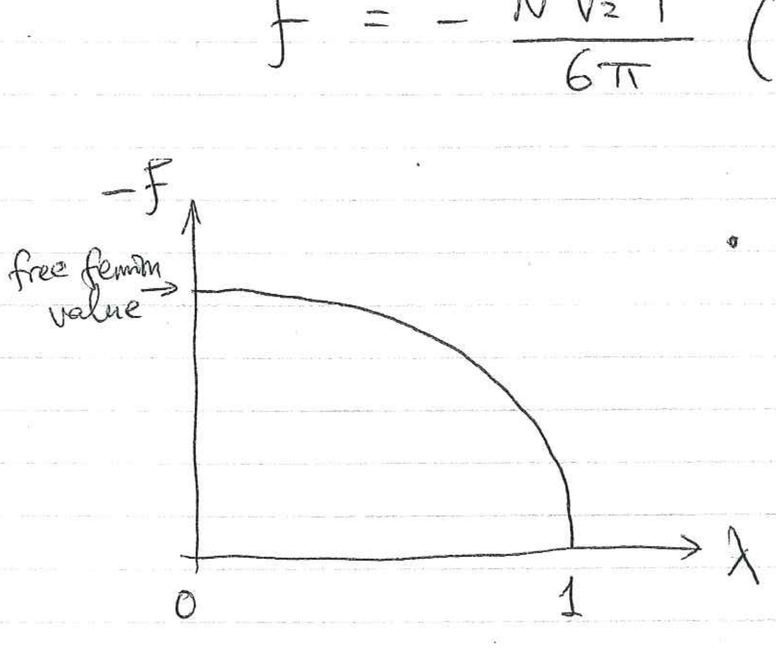
precise form depended on $\bar{\psi}\gamma_\mu A^\mu \psi$ coupling

(different rel'n to self-energy for

CS-scalar th.)

Result:

$$F = -\frac{N V_2 T^3}{6\pi}\left(c^{\frac{3}{2}}\frac{1-\lambda}{\lambda} + 6\int_{\sqrt{c}}^{\infty} dy \cdot y \ln(1+e^{-y})\right)$$



- CFT does not exist for $\underbrace{\lambda > 1}$.

  ↖ a regularization scheme dependent statement.

- $\lambda = \frac{N}{k}$, dim'l reduction.
- if use YM regularization, $k_{YM} CS(A) + \int \mathrm{tr} F_{\mu\nu}F^{\mu\nu}$

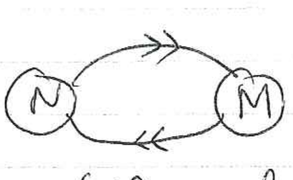
  $\Rightarrow$ 1-loop shift $k_{YM} \to k_{YM} + N$.

  $0 < \lambda = \frac{N}{k_{YM}+N} < 1$, consistent w/ our finding.

- A Supersymmetric CS vector Model and Connecting to Strings.

$\mathcal{N}=6$ ABJ theory

$$U(N)_k \times U(M)_{-k} \xrightarrow{M=1} U(N)_k \times U(1)_{-k}$$



($\mathcal{N}=2$ multiplets)

$\downarrow$ integrate out $U(1)\times U(1)$

$$SU(N)_k + \begin{matrix}\square \\ \overline{\square}\end{matrix} \text{ matter}$$

$\nearrow$

a CS-vector model!

- Dual to a HSGT in AdS_4.
  - bulk classical eqn's must have HS <u>gauge sym</u>, so that no new longitudinal d.o.f. arise.
  - parity violating (Vasiliev theory? susic)

$U(N)_k \times U(M)_{-k}$ ABJ.

$\updownarrow$

$U(M)$ - matrix $\mathcal{N}=6$ Vasiliev theory in AdS_4 ~~parity~~

- turn on mixed bdry condition

for bulk $SU(M)$ vector gauge fields.

$$\frac{1}{2}\epsilon_{ijk}F_{jk} + \frac{N}{k}F_{zi}\Big|_{z=0} = 0.$$

SU(M) field strength.

U(M)$_{-k}$

bulk coupling $g^2 \sim \frac{1}{N}$.

BULK 't Hooft coupling

$$\lambda_{BULK} = g^2 M = \frac{M}{N} \; !$$

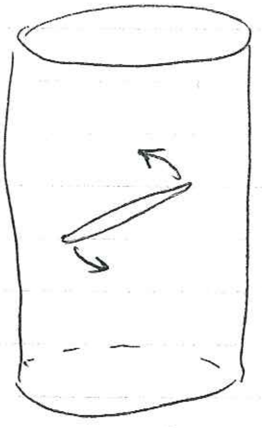
Single trace operator wrt $U(N)$,

HS currents.

$$\Delta = S + 1 + O\left(\frac{M}{N}\right).$$

GKP scaling

$\left(\#\frac{M}{N} + \#\left(\frac{M}{N}\right)^2 + \cdots\right) \ln S$ for $S \gg 1$.

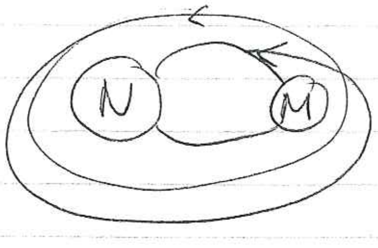


Spinning folded semi-classical string in AdS_4

$$\Delta - J \sim \left(\frac{R}{l_s}\right)^2 \ln J , \quad J \gg 1 .$$

HS fields $\longrightarrow$ (spinning) strings

$\frac{M}{N} \ll 1$, $\quad \frac{M}{N} \sim 1 \quad M/N$



single-trace operators of the quiver $\longleftrightarrow$ single string states

$\Big\downarrow$ small M.

multi-particle states of HS fields.

A generic string mode is the bound state of HS particles in AdS_4 !

binding energy $\sim \mathcal{O}\left(\frac{M}{N}\right)$.

ABJ is known to be dual to

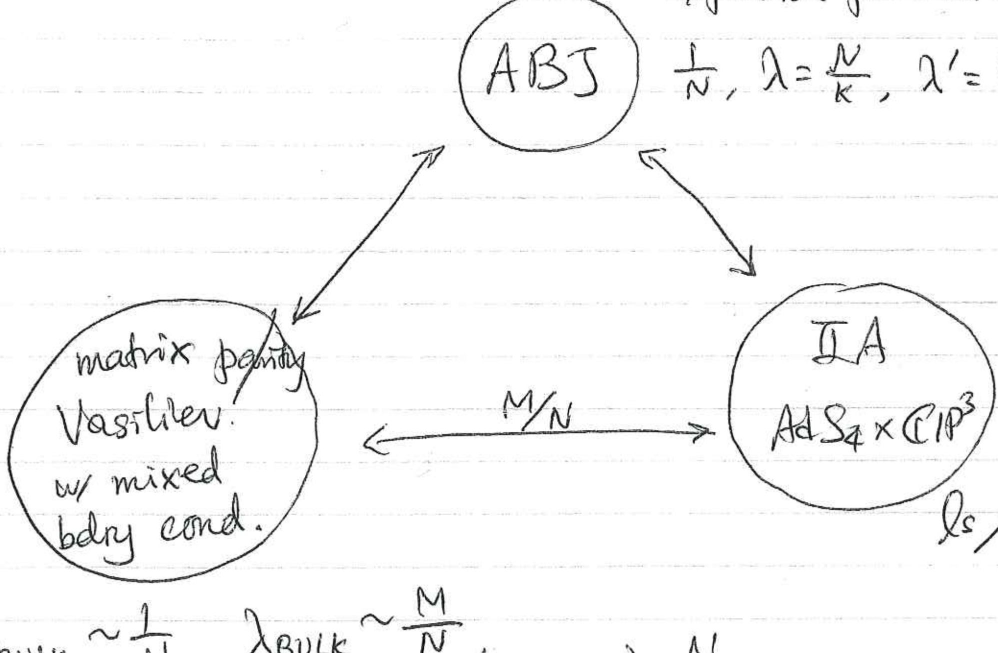
Type IIA string on

$AdS_4 \times \mathbb{CP}^3$ w/ flat B-field.

$$\frac{1}{2\pi\alpha'}\int_{\mathbb{CP}^1} B = \frac{N-M}{k} + \frac{1}{2}.$$

$\frac{|N-M|}{k} < 1$.

expansion parameter



$\frac{1}{N}$, $\lambda = \frac{N}{k}$, $\lambda' = \frac{M}{k}$

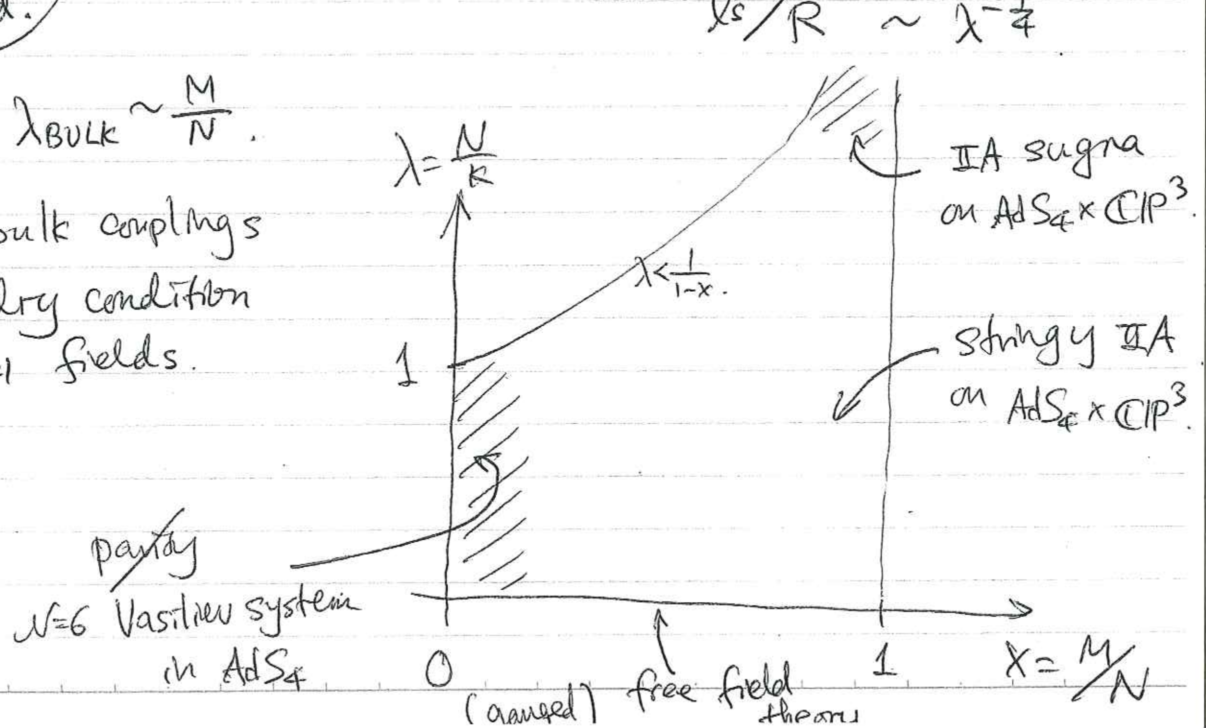
$g_s \sim \frac{1}{N}$,

$B \sim \frac{N-M}{k}$.

$l_s/R \sim \lambda^{-\frac{1}{4}}$

$g_{BULK} \sim \frac{1}{N}$, $\lambda_{BULK} \sim \frac{M}{N}$.

$\lambda = \frac{N}{k}$ enters bulk couplings and bdry condition on $s=1$ fields.



A story I had left out entirely.

AdS_3/CFT_2.

pure HS

$SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ CS. $\longleftrightarrow$ HS "extremal" CFT? don't know

$s=2,3,\ldots,N$

Vasiliev system

$hs[\lambda] \times hs[\lambda]$ $\longleftrightarrow$ a subsector of W_N min model

$s=2,3,\ldots,\infty$

+ massive scalar φ

- primary $(R;0)$
- OPE closed at large N
- incomplete at finite N.

Vasiliev system

+ more massive scalars $\longleftrightarrow$ W_N minimal model.

+ many more light states.

- large N factorization ✓.

- Conical deficits?

$$\frac{SU(N)_k \times SU(N)_1}{SU(N)_{k+1}} \qquad \lambda = \frac{N}{k+N}.$$